PRACTICE FINAL SOLUTIONS
MATH 225

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Problem 1. *Do all Exercise 1 in all sections!*

Problem 2.

Notice that $v_1 - v_2 = e_3$, $v_1 - v_3 = e_2$, and so $v_1 - e_2 - e_3 = v_2 + v_3 - v_1 = e_1$. We could compute the matrix directly, but we prefer to do so using the change of basis matrix from $\beta = \{e_1, e_2, e_3\}$ to $\gamma = \{v_1, v_2, v_3\}$ as we have just shown that this is indeed a basis. We see that $Q = [I]_\beta^\gamma = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ and $Q^{-1} = [I]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. We may now calculate $[T]_\beta = Q^{-1}[T]_\gamma Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}$.

Problem 3.

The characteristic polynomial of A is $\det(A - tI) = \det \begin{pmatrix} -\sqrt{3}/2 - t & -1/2 \\ 1/2 & -\sqrt{3}/2 - t \end{pmatrix} = (\sqrt{3}/2 + t)^2 + 1/4 = (t^2 + \sqrt{3}t + 3/4) + 1/4 = t^2 + \sqrt{3}t + 1$. This doesn't have real roots, so there are no real eigenvalues.

However, considering A as a complex matrix (in particular, computing A^{25} doesn't depend on whether you think of A as real or complex). Then over the complex numbers, this polynomial splits. Plugging into the quadratic equation, we see the roots of this polynomial are $\frac{-\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{-\sqrt{3} \pm i}{2}$. Thus A has eigenvectors with these eigenvalues, and as there are two and A is two dimensional, these must be all of them. We solve for the eigenvectors:

$$A - \frac{-\sqrt{3} + i}{2} I = \begin{pmatrix} -\sqrt{3}/2 - (\sqrt{3}/2 + i/2) & -1/2 \\ 1/2 & -\sqrt{3}/2 - (\sqrt{3}/2 + i/2) \end{pmatrix} = \begin{pmatrix} -i/2 & -1/2 \\ 1/2 & -i/2 \end{pmatrix} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}.$$

This matrix is singular, as it should be, so the only relation is given by the top row, $ix + y = 0$, i.e. $x = iy$. An eigenvector is thus $(i, 1)$. Similarly, an eigenvector for the

Date: April 29, 2014.

other eigenvalue $\frac{-\sqrt{3}-i}{2}$ is seen to be $(i,-1)$. So the basis $\gamma = \{v_1, v_2\} = \{(i,1),(i,-1)\}$ diagonalizes A. Letting $Q = [I]_\gamma^\beta = \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$, $Q^{-1} = \begin{pmatrix} -i/2 & 1/2 \\ -i/2 & -1/2 \end{pmatrix}$. Therefore we have

$$Q^{-1}AQ = \begin{pmatrix} -i/2 & 1/2 \\ -i/2 & -1/2 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{-\sqrt{3}+i}{2} & 0 \\ 0 & \frac{-3-i}{2} \end{pmatrix} = D.$$

Finally, notice that $\frac{-\sqrt{3}+i}{2} = e^{5\pi i/6}$ and $\frac{-\sqrt{3}+i}{2} = e^{7\pi i/6}$. Therefore, raised to the 24th power they are 1. So, $D^{24} = I$, so $D^{25} = D$, and therefore $A^{25} = (QDQ^{-1})^{25} = QD^{25}Q^{-1} = QDQ^{-1} = A$.

Problem 4.

This is currently under construction.

Problem 5.

We see that the inner product of the first and fourth columns is $\frac{1}{\sqrt{6}} \neq 0$, so the matrix is not orthogonal. Normal means it commutes with its adjoint, i.e. its conjugate transpose. In this case all entries are real, so this is just its transpose. $BB^* = \big(< R_i, R_j >\big)$ where R_i is the ith row of B. $B^*B = \big(< C_i, C_j >\big)$, where C_i is the ith column of B. By inspection $13/12 =< R_1, R_1 > \neq < C_1, C_1 >= 1$, so B is not normal.

Problem 6.

We apply Gram-Schmidt. Let the above vectors be v_1, v_2, v_3 respectively. Let $u_1 = v_1$. Then $u_2 = v_2 - \frac{<v_2,v_1>}{||v_1||^2} v_1 = (-1,4,4,-1) - 6/4(1,1,1,1) = (-5/2, 5/2, 5/2, -5/2)$. Noting that multiplying by constants doesn't affect orthogonality, we replace u_2 with $(-1,1,1,-1)$. Now $u_3 = v_3 - \frac{<v_3,u_1>}{||u_1||^2} u_1 - \frac{<v_3,u_2>}{||u_2||^2} u_2 = (4,-2,2,0) - \frac{4}{4}(1,1,1,1) - \frac{-4}{4}(-1,1,1,-1) = (2,-2,2,-2)$. Normalizing these, we obtain an orthonormal basis $\{u_1', u_2', u_3'\} = \{(1/2,1/2,1/2,1/2), (-1/2,1/2,1/2,-1/2), (1/2,-1/2,1/2,-1/2)\}$.

Problem 7.

We eliminate the xy-term by diagonalizing $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$. This satisfies $4x^2 + 4xy + 4y^2 = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}$. This has characteristic polynomial $(4-t)^2 - 4 = t^2 - 8t + 12 = (t-6)(t-2)$. So its eigenvalues are 6 and 2. Let's find the eigenvectors: $A - 6I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, so we need $x - y = 0$, i.e. $x = y$, so $v_1 = (1/\sqrt{2}, 1/\sqrt{2})$ is a unit eigenvector. Similarly, $A - 2I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ clearly has kernel $x - y = 0$, so $v_2 = (1/\sqrt{2}, -1/\sqrt{2})$ is a unit eigenvector. Now let $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then $P^tAP = D = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$. Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and let $X' = P^tX$. Then $X^tAX = X^t(PDP^t)X = X'^tDX' = 6x'^2 + 2y'^2$, where $X' = \begin{pmatrix} x' \\ y' \end{pmatrix}$. $P^tX = 1/\sqrt{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1/\sqrt{2}\begin{pmatrix} x+y \\ x-y \end{pmatrix}$. So after this change of basis, we have $6x'^2 + 2y'^2 = 1$, a standard form for an ellipse (i.e. $(x/a)^2 + (y/b)^2 = 1$ with $a = 1/\sqrt{6}$, $b = 1/\sqrt{2}$).