Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2014

Problem Set # 1 (due 4 pm Friday 24 January 2014)

Notation: If S is a set of elements (numbers, vectors, rabbits, ...) then the notation "$s \in S$" means "s is an element of the set S." If T is another set, then the notation "$T \subseteq S$" means "every element of T is an element of S" or "T is a **subset** of S." For example, the set of squares is a subset of the set of rectangles.

We have notations for the following commonly referred to sets:

- $\mathbb{Z}$ is the set of integers (i.e., whole numbers, positive or negative).
- $\mathbb{Q}$ is the set of rational numbers (i.e., fractions $\frac{a}{b}$ for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $b \neq 0$). It is a field, see Appendix C.
- $\mathbb{R}$ is the set of real numbers. It is a field, see Appendix C.
- $\mathbb{C}$ is the set of complex numbers (i.e., $a + bi$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}$, where $i^2 = -1$). It is a field, see Appendix C.
- If F is any field then F^n is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where each $a_i \in F$ for $i = 1, \ldots, n$. It is a vector space over F, see FIS 1.2 Example 1.

If S and T are sets, then a **function** $f : S \to T$ **from** S **to** T is the a rule that associates to each element $s \in S$, an element $f(s) \in T$. For example, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$, is a function. Another example, if S is the set of people in the room, $f : S \to \mathbb{Z}$ assigning to each person $p \in S$, their height $f(p) \in \mathbb{Z}$ in inches rounded up to the nearest inch, is a function.

Let S be a set and F be a field. Define $\mathcal{F}(S, F)$ to be the set of all functions $f : S \to F$. Then $\mathcal{F}(S, F)$ is a vector space over F by FIS 1.2 Example 3. The set of polynomials $\mathsf{P}(F)$ with coefficients in F is a vector space over F by FIS 1.2 Example 4. In fact, $\mathsf{P}(F) \subset \mathcal{F}(F, F)$ is a subspace. For each $n \geq 0$, the set of polynomials $\mathsf{P}_n(F)$ of degree at most n and with coefficients in F is also a vector space over F, and a subspace of $\mathsf{P}(F)$.

Reading: FIS 1.1–1.3

Problems:

1. FIS 1.2 Exercises 1, 9 (Hint: To prove that any zero vector is unique, suppose that 0 and $0'$ are zero vectors and then show using the zero vector axioms that $0 = 0'$), 10, 13, 17.

2. FIS 1.3 Exercises 1, 8abcf, 10, 11, 12, 20.

3. Prove that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Prove that $\mathbb{R}$ is a vector space over $\mathbb{Q}$. (The scalar multiplication in both cases is regular multiplication of numbers. You can use the facts that $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields, as mentioned in Appendix C and D.)

As an aside, we don't yet know formally about "dimension", but you'll see that $\mathbb{C}$ has dimension 2 over $\mathbb{R}$ while $\mathbb{R}$ is actually infinite dimensional over $\mathbb{Q}$.

4. Prove that the set $\mathbb{Q}(\sqrt{2})$, of real numbers of the form $a + b\sqrt{2}$ for $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$, is a field. (Hint: The most important field axiom you must verify is that every nonzero element of $\mathbb{Q}(\sqrt{2})$ has a multiplicative inverse.)

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