Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2014

Midterm 2 Review Sheet

Directions: The second midterm exam will take place during lecture on Tuesday, April 1st. You will have the entire class period, 1 hour and 15 minutes, to complete the exam. No electronic devices will be allowed. No notes will be allowed. On all problems, you will have to show your work to get full credit.

Topics covered and practice problems:

- Matrix multiplication and composition of linear maps. Left multiplication maps. FIS 2.3 Exercises 3, 4, 9, 11, 12.

- Invertibility of linear maps. Isomorphism. Classification of finite dimensional vector spaces. FIS 2.4 Exercises 1, 2, 3, 5, 9, 15, 16, 19.

- Change of coordinate matrix. FIS 2.5 Exercises 2, 3, 4, 5, 6.

- Elementary row and column operations. Elementary matrices. FIS 3.1 Exercises 2, 3, 5, 8.

- Matrix rank. Matrix inverse. FIS 3.2 Exercises 2, 3, 4, 5, 6, 7, 11, 19, 20, 21, 22.

- Systems of linear equations $Ax = b$. Homogeneous and inhomogeneous systems. Systems with a unique solution. Consistent and inconsistent systems. FIS 3.3 Exercises 2, 3, 4, 5, 7, 8, 9.

- Gaussian elimination. Reduced row echelon form and interpretation. Finding a basis in a generating set. Extending a linearly independent set to a basis. FIS 3.4 Exercises 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.

- Determinants. Determinants detect linear independence. FIS 4.1 Exercises 3, 7, 8, 10. FIS 4.2 Exercises 2, 3, 4, 5–22, 23, 25, 26, 27, 28, 29, 30. FIS 4.3 Exercises 2–7, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 23, 24. FIS 4.4 Exercises 2, 3, 4.

Practice exam:

1. Find a cubic polynomial $p(x)$ with real coefficients such that $p(-2) = 0$, $p(-1) = 4$, $p(1) = 0$, and $p(2) = 4$.

2. Consider the system of linear equations with real coefficients:

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 3y + z &= 3 \\ x + 3y + az &= 0 \\ x + y + 2z &= b \end{aligned}$$

i) Find values of the parameters a and b for which the system has no solution.
ii) Find values of the parameters a and b for which the system has a unique solution, and find this solution.
iii) Find values of the parameters a and b for which the system has infinitely many solutions, and find all solutions of the resulting system.

3. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Calculate $Q^{-1}AQ$. (Hint: You don't need to do any matrix multiplications!)

4. Calculate the ranks of the following real matrices:

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 8 \\ 2 & 4 & 0 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 3 & 6 \\ 1 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 40 \\ 3 & 0 & 6 & 7 & 15 \end{pmatrix}$$

5. If possible, find a subset of the vectors $(1, 4, -2)$, $(-3, -5, 8)$, $(1, 37, -17)$, $(-8, 12, -4)$, and $(2, -3, 1)$ that are a basis for $\mathbb{R}^3$. If not, prove that it is impossible.

Extend $(2, 1, 3, 4)$, $(1, 1, 1, 3)$ to a basis of $\mathbb{R}^4$.

6. Find the determinants of the following matrices with real entries:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 7 & 9 & 11 & 13 \\ 0 & 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 8 & 9 & 10 & 11 \\ 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Yale University, Department of Mathematics, 10 Hillhouse Ave, New Haven, CT 06511
E-mail address: asher.auel@yale.edu