

Problem Set # 4 (due Thursday 16 February)

**Notation:** Let  $F$  be a field. For  $A \in M_{n \times n}(F)$  a square matrix, define the **trace**  $\text{tr}(A) \in F$  to be the sum of the diagonal entries of  $A$ , i.e., if  $A = (a_{ij})$  then  $\text{tr}(A) = a_{11} + \cdots + a_{nn}$ . You can check that  $\text{tr} : M_{n \times n}(F) \rightarrow F$  is actually a linear map of  $F$ -vector spaces.

Let  $V$  be an  $F$ -vector space,  $T : V \rightarrow V$  a linear map, and  $W \subset V$  a subspace. Denote by  $T|_W : W \rightarrow V$  the **restriction** of  $T$  to  $W$ , i.e.,  $T|_W(w) = T(w)$  for every  $w \in W$ . Then  $W$  is called  **$T$ -invariant** if  $R(T|_W) \subset W$ , i.e.,  $T$  takes vectors from  $W$  to vectors in  $W$ . For example,  $\{0\} \subset V$  and  $V \subset V$  are always  $T$ -invariant subspaces for any linear map  $T : V \rightarrow V$ . As another example, the  $x$ -axis is  $T$ -invariant for the linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T(x, y) = (x + y, y)$ .

Let  $W_1$  and  $W_2$  be  $F$ -vector spaces. Define the **cartesian product**  $W_1 \times W_2$  to be the set of ordered pairs  $(w_1, w_2)$  of vectors  $w_1 \in W_1$  and  $w_2 \in W_2$ . In fact,  $W_1 \times W_2$  is an  $F$ -vector space under component-wise addition and scalar multiplication. For example,  $F^2 = F \times F$ .

**Reading:** FIS 2.1

**Problems:**

1. FIS 2.1 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 3, 5, 9, 11, 12, 16, 17, 18, 21.

2. This problem is intentionally left blank.

3. Let  $W_1 \subset V$  and  $W_2 \subset V$  be subspaces of an  $F$ -vector space  $V$ . Consider the map

$$\Sigma : W_1 \times W_2 \rightarrow W_1 + W_2$$

defined by  $\Sigma(v, w) = v + w$ . Verify that  $\Sigma$  is a linear map and is surjective. Prove that  $W_1 + W_2 = W_1 \oplus W_2$  if and only if  $\Sigma$  is injective.

In that case,  $\Sigma : W_1 \times W_2 \rightarrow W_1 \oplus W_2$  is bijective (i.e., injective and surjective), so you can think of the subspace  $W_1 \oplus W_2$  “internal” to  $V$  as an object  $W_1 \times W_2$  “external” to  $V$ .

4. Let  $T : F^2 \rightarrow F^2$  be the linear transformation defined by  $T(x, y) = (-y, x)$ . For each choice of  $F = \mathbb{F}_2, \mathbb{F}_p$  for  $p$  odd,  $\mathbb{R}$ , and  $\mathbb{C}$ , find all  $T$ -invariant subspaces of  $F^2$ .