Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2017

Problem Set # 6 (due in class Thursday March 2nd)

Reading: FIS 2.4, 2.5, 3.1

Problems:

1. FIS 2.4 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 9 (By "arbitrary matrices" they mean matrices that are not necessarily square), 10 (The definition of a matrix A being invertible is that there exists B such that both AB and BA are the identity, in this problem you prove that you only need to know one of these), 15, 17 (For the first part, use the restriction of T to V_0; for the second part, use exercise 15), 20.

2. FIS 2.5 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2d, 3b, 6bd, 13.

3. FIS 3.1 Exercises 1 (If true, cite or prove it; if false, give a counterexample), 3c, 9.

4. Let F be a field and consider the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $M_{2\times 2}(F)$.

(a) Prove that M is invertible if and only if $ad - bc \neq 0$, in which case

$$M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(Hint. Use the Rank–Nullity theorem and work you did on Problem Set # 2.)

(b) Write down all invertible matrices in $M_{2\times 2}(\mathbb{F}_2)$.

5. Let V be an $\mathbb{F}_p$-vector space of dimension n.

(a) Calculate the number of vectors in V. This is a function of p and n.
(Hint: You can you assume that $V = \mathbb{F}_p^n$. Why?)

(b) For each $1 \leq k \leq n$, calculate the number of k-tuples of linearly independent vectors in V. This is a function of p, n, and k.
(Hint: Start with a non-zero vector v_1, then choose v_2 not in the span of $\{v_1\}$, then choose v_3 not in the span of $\{v_1, v_2\}$, and so on, using the fact that you know the size of the span by the previous part.)

(c) Calculate the number of invertible $n \times n$ matrices over $\mathbb{F}_p$.
(Hint: Use the previous part.)

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