

Problem Set # 9 (due in class Thursday April 13th)

Notation: Let $g(t)$ be a polynomial with coefficients in a field F . Let $T : V \rightarrow V$ be a linear operator on an F -vector space V . Then we define a linear operator $g(T) : V \rightarrow V$ as follows. First, for an integer $n \geq 1$, denote by $T^n : V \rightarrow V$ the n -fold composition

$$T^n = \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}}$$

of T with itself. We define $T^0 = I_V$. Writing

$$g(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0,$$

then “ g applied to the linear operator T ” is defined to be

$$g(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 I_V,$$

where the scalar multiples and sums are being taking inside the F -vector space $\mathcal{L}(V)$ of linear operators on V .

For an $n \times n$ matrix A with coefficients in F , there is an analogous definition of $g(A)$. Make sure you understand the content of the equality of linear operators $g(L_A) = L_{g(A)}$ on F^n .

Let $f(t)$ be the characteristic polynomial of a matrix $A \in M_{n \times n}(F)$. Assume that $f(t)$ splits (i.e., is a product of linear factors) as

$$f(t) = (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_r)^{m_r}$$

where $\lambda_1, \dots, \lambda_r \in F$ are the eigenvalues of A . We call m_i the **multiplicity** of the eigenvalue λ_i .

Reading: FIS 5.1–5.2

Problems:

1. FIS 5.1 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 3bd, 4e, 8, 9, 10, 11, 14, 17, 22, 23.

Think about, but do not hand in: 2, 15, 16, 19, 20.

2. FIS 5.2 Exercises 9, 10, 11 (Hint: The easiest way to do part (a) is to do FIS 5.1 Exercise 16, which in turn means doing FIS 2.3 Exercise 13).

Think about, but do not hand in: 2, 3, 4, 12.