PRACTICE MIDTERM #1

Exercise 1

(1), (2) S_1, S_2 are the null spaces of the maps $T : V \to \mathbb{R}$ given by $T(f) = f(a)$ where $a = 0, 1$ respectively. As the rank of this map is 1 (the constant polynomials map onto $\mathbb{R}$, for example), and the dimension of V is 4, by the Rank-Nullity Theorem, the dimension of S_1 and S_2 is three.

(3) S_3 is not a subspace as it does not contain 0, for example.

(4) S_4 is a subspace, being the intersection of the subspaces S_1 and S_2. It is of dimension two: $\{x(x-1), x^2(x-1)\}$ is a basis.

(5) S_5 is not a subspace, since it is the union of two subspaces ($S_5 = S_1 \cup S_2$) and neither contains the other (to refer to an exercise many of you have seen in recitation). Directly, we have $x \in S_5$ and $x - 1 \in S_5$ but $f(x) = x + (x-1) = 2x - 1 \notin S_5$, as $f(0) = -1$ and $f(1) = 1$.

(6) S_6 is a subspace, as it is the null space of the linear map $T : V \to V$ given by $T(f) = f(0) + f(1)$ (you should check that this is a linear map).

(7) S_7 is a subspace, since the equation $p(0)^2 + p(1)^2 = 0$ is equivalent to $p(0) = 0$ and $p(1) = 0$ (so in fact, $S_7 = S_4$). Since $p(0)$ and $p(1)$ are real numbers, their squares are nonnegative, and hence the sum of their squares can only be zero when both are zero.

Exercise 2

(1) The subset $\{(0,1,3), (1,2,3), (2,3,1)\} = \{v_1, v_2, v_3\}$ is a basis. To prove this, first notice by inspection that v_1 and v_2 are not scalar multiples of each other, and are nonzero, so $\{v_1, v_2\}$ is a linearly independent set. Now, in order to show that we may add v_3 to this set without losing linear independence, it is sufficient to show that $v_3 \notin \text{span}(\{v_1, v_2\})$. Suppose by contradiction that we have $v_3 = av_1 + bv_2$, i.e. $v_3 \in \text{span}(\{v_1, v_2\})$. Then equating coordinates, we have $2 = b, 3 = a + 2b$, and $1 = 3a + 3b$. Substituting $b = 2$ in the second equation yields $3 = a + 4 \Rightarrow a = -1$, and plugging both of these into the third equation yields $1 = 3(-1) + 3(2) = 3$, a contradiction in the field $\mathbb{R}$. Therefore the set $\{v_1, v_2, v_3\}$ is linearly independent. As it has size three, and we know the dimension of $\mathbb{R}^3$ is 3, it is a basis.

To express $(3,3,3)$ in terms of this basis, we solve the equations $a(0,1,3) + b(1,2,3) + c(2,3,1) = (3,3,3)$, i.e. $b + 2c = 3, a + 2b + 3c = 3, 3a + 3b + c = 3$. Subtracting the second from the first, we get $a + b + c = 0$. Subtracting three times this from the third equation, we find $-2c = 3$, so $c = -3/2$. The original first equation $b + 2c = 3$ now gives $b = 6$, and the equation $a + b + c = 0$ gives $a = -9/2$.

(2) To extend this linearly independent set to a basis, it suffices to add any vector outside of $\text{span}(S_2)$. Notice that the first two coordinates of each element in S_2 are equal. Thus any linear combination of elements in S_2 will have the first two coordinates equal. So, the vector $(1,0,0)$, for example, does not lie in $\text{span}(S_2)$, and so extends S_2 to a basis. By inspection, $(3,3,5) = (1,1,1) + 2(1,1,2)$.

Exercise 3

$N(T) = \{f \in \mathsf{P}_2(\mathbb{R}) : (x-1)f = 0\} = \{0\}$, as the degree of $(x-1)f(x)$ is at least one unless $f(x) = 0$. Therefore T is one-to-one and has nullity$(T) = 0$ and rank$(T) = 3$. This implies that the images of vectors forming a basis for $\mathsf{P}_2(\mathbb{R})$ will be a basis for the range (they will generate, and there are the correct number of them). Therefore $\{T(1), T(x), T(x^2)\} = \{(x-1), (x-1)x, (x-1)x^2\}$ is a basis for the range of T.

Exercise 4

(1) If e^x and xe^x were linearly dependent, there would exist some scalar $c \in \mathbb{R}\backslash\{0\}$ such that $cxe^x = e^x$. As the right-hand side is always positive, and the left-hand side can be negative regardless of the value of c, this is impossible.

(2) Let $f(x) = ae^x + bxe^x$ be an element of V. Then $f(x) \in N\left(\frac{d}{dx}\right)$ if and only if

$$\frac{df}{dx} = ae^x + be^x + bxe^x = 0 = (a+b)e^x + bxe^x = 0.$$

Since e^x and xe^x are linearly independent, we see that $f(x) \in N\left(\frac{d}{dx}\right)$ if and only if $a + b = 0$ and $b = 0$. As the only solutions are $a = b = 0$, we have that $N\left(\frac{d}{dx}\right) = 0$. Once again, $\frac{d}{dx} : V \to V$ is one-to-one, has nullity 0, rank 2, and a basis for the range is given by the images of the basis vectors of the domain, i.e. $\{e^x, e^x + xe^x\}$. But since span$(\{e^x, e^x + xe^x\})$=span$(\{e^x, xe^x\}) = V$, we see that this map is also onto.

(3) As we have $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}xe^x = e^x + xe^x$, the matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Exercise 5

Let $\varepsilon = \{e_1, e_2, e_3, e_4, e_5\}$ be the standard ordered basis of $\mathbb{R}^5$ and let $\gamma = \{\gamma_1, \cdots, \gamma_5\} = \{e_2, e_4, e_5, e_1, e_3\}$ be a different ordered basis, just given by a permutation of the basis vectors. Then Q is the change of basis matrix $[I]_\gamma^\varepsilon$, and so $Q^{-1}AQ$ will simply be $[L_A]_\gamma$ (see Theorem 2.23 and its Corollary).

So we need to compute the matrix representation $[L_A]_\gamma$. For simplicity of notation, let $v = e_1 + e_2 + e_3 + e_4 + e_5$ (represented by the vector of all 1's). Then $A(\gamma_1) = A(e_2) = v - e_3 = v - \gamma_5$, so the first column of $[L_A]_\gamma$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Similarly we compute $[L_A]_\gamma = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.