Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2018

Problem Set # 7 (due in class Thursday March 29th; have a great Spring break!)

Notation: Let $A \in M_{m\times n}(F)$ and $b \in F^m$. A system of linear equations $Ax = b$ is called **consistent** if there is at least one solution, and **inconsistent** if there are no solutions.

Reading: FIS 3.1, 3.2, 3.3, 3.4. This problem set is mostly practice solving systems of linear equations!

Problems:

1. FIS 3.2 Exercises 1 (If true, cite or prove it; if false, give a counterexample), 2bdf, 14, 17.

2. FIS 3.3 Exercises 1 (If true, cite or prove it; if false, give a counterexample), 4, 7ae, 8.

3. FIS 3.4 Exercises 1 (If true, cite or prove it; if false, give a counterexample), 2bf, 3 (the condition can be succinctly restated as "$(A'|b')$ contains a pivot in the last column"), 4c, 9.

4. *Balancing a chemical reaction.* Consider the chemical reaction producing *nitrous oxide* gas

$$a\mathrm{NH}_3 + b\mathrm{O}_2 \to c\mathrm{N}_2\mathrm{O} + d\mathrm{H}_2\mathrm{O}$$

where a, b, c are unknown positive integers. The reaction must be balanced so that the number of atoms of each element must be the same before and after the reaction. For example, considering the number of *oxygen* atoms, we get the equation $2b = c + d$. Chemists like to have the smallest possible positive integer values of a, b, c that balance the reaction. Balance this reaction.

5. A single player game is played on a 2×2 grid of boxes. At the start of the game, the boxes are colored randomly either black or white. During each turn, the player selects a box. Upon selecting a box, all *other* boxes change color. The player's objective is to make all the boxes white.

Since there are *only* 16 starting positions, you could analyze each one. But why would you want to do that when you could use linear algebra!?

(a) Prove that no matter what the starting position, the game is always winnable.

(b) Which starting positions have the longest minimal sequence of moves to win?

(c) Consider the same game, except with three colors, (say white, black, and red), and selecting a box causes the *other* boxes to cycle one color (white to black to red to white). Prove that this game is *not* always winnable! Write down an initial starting position that is not winnable.

(d) Prove that starting with a winnable position (in either the two color or the three color game), the order in which the player presses the boxes is irrelevant, only the number of times each box needs to be pressed is relevent. (This sounds surprising!)

(e) What if there are 5 colors instead of 3, is the game always winnable?

Yale University, Department of Mathematics, 10 Hillhouse Ave, New Haven, CT 06511
E-mail address: asher.auel@yale.edu