Yale University Department of Mathematics
Math 235 Reflection Groups
Spring Semester 2016

Instructor:	Professor Asher Auel	**Lecture:**	LOM 205
Office:	LOM 210	**Time:**	Tue Thu 1:00 – 2:15 pm

Text:	*Finite Reflection Groups*, 2nd Ed., Grove and Benson Springer-Verlag, 2010. ISBN-13: 978-1-4419-3072-9.
Web-site:	http://math.yale.edu/~auel/courses/235s16/

Introduction: A reflection is a linear transformation describing the mirror image about a line in the plane, about a plane in 3-dimensional space, or generally, about a hyperplane in n-dimensional space. A reflection group is a discrete group generated by reflections. These include symmetry groups of regular polytopes (e.g., Platonic solids) and of regular tilings (e.g., wallpaper patterns). A kaleidoscope is an example of an optical device for visualizing certain reflection groups in the plane. The theory of reflection groups links linear algebra, abstract algebra (particularly group theory), Lie algebras, and representation theory in a beautiful way.

The main topics covered will be orthogonal transformations, reflections in real Euclidean space, Coxeter groups, crystallographic groups, root systems, and the classification of finite Coxeter groups. Along the way, the course will cover the basics of (finite) group theory as well as certain advanced topics in linear algebra. This will be a heavily proof-based course with homework requiring a significant investment of time and thought. This course is only appropriate to students who have taken a first course in linear algebra. A previous course in abstract algebra is not necessary. While the course is primarily targeted at students interested in studying higher mathematics, the subject matter would be of interest (and possible use) in subjects such as chemistry, computer science, materials science, and theoretical physics.

Grading: Your final grade will be calculated according to the table at right. Notice that more overall emphasis is placed on exams than on weekly homework assignments. On the other hand, completing your weekly homework will be crucial to your success on the exams.

Homework	20 %
Midterm (Thu 31 Mar)	30 %
Final Exam (Sun 8 May)	50 %

Exams: The midterm exam will take place in class on Thursday 31 March 2016. The final exam will take place 2:00 – 5:30 pm on Sunday 8 May 2016. Make-up exams will only be allowed with a dean's excuse. The use of electronic devices during exams will not be allowed. Unless otherwise stated, you will always need to show your work and write coherent mathematical sentences.

Homework: You will be assigned a weekly problem set, due in my departmental mailbox on Thursday by 4 pm. The problem sets will be posted on the course web-site syllabus page the week before they are due. Your lowest problem set score will be dropped from your final grade calculation. *Late or improperly submitted homework will not be accepted.* If you know in advance that you will be unable to submit your homework at the correct time and place, you must make special arrangements ahead of time. Under extraordinary circumstances, late homework may be accepted with a dean's excuse.

Consider (as you would for any other class) the pieces of paper you turn in as a final copy: written neatly and straight across the page, on clean paper, stapled together, with nice margins, lots of space, and well organized. *If it's not readable, it won't be graded.* You should strongly consider starting with a rough draft, especially on problems requiring a proof. You will need to show your work on computational problems. You might consider taking the opportunity to learn LaTeX.

Group work, honestly: Working with other people on mathematics is not only allowable, but is highly encouraged and fun. You may work with anyone (e.g., other students in the course, students not in the course, tutors, bums on the street) on the rough draft of your problem sets. If done right, you'll learn the material better and more efficiently working in groups. The golden rule is:

> you may work with anyone on *solving* your homework problems,
> but you must *write* up your final draft by yourself.

Writing up the final draft is as important a process as figuring out the problems on scratch paper with your friends. Mathematical writing is very idiosyncratic—it is easy to tell if papers have been copied—just don't do it! You will not learn by copying solutions from others or from the internet! Also, if you work with other people on solving a particular problem, you *must list your collaborators at the top of the paper*, as well as any resources (e.g., Wikipedia) used beside the text book. Make the process fun, transparent, and honest.

Prerequisites: The official prerequisite is linear algebra, either Math 222 or 225. Specifically, you should know about abstract vector spaces, bases, linear transformations, matrix representations, and eigenvalues/eigenvectors. The course will include a brief review of inner product spaces. The unofficial prerequisites are a mature mathematical mind, some experience with writing proofs, and the desire to work hard.

Topics covered: Subject to change.

(1) Regular polygons and regular polyhedra. Orthogonal matrices. Orthogonal transformations in 2-dimensional Euclidean space. Rotations and reflections in the plane. Finite subgroups of $O(2)$. GB 2.1–2.2.

(2) Review of inner products, Euclidean space, groups, subgroups, cyclic groups, cosets, normal subgroups, quotients, Lagrange's theorem, homomorphisms, group actions, orbits and stabilizers. GB 1.1–1.2.

(3) Orthogonal transformations in 3-dimensional Euclidean space. Structure of rotations and reflections in space. Symmetry groups of regular polyhedra. Finite subgroups of $O(3)$. Crystallographic groups. GB 2.1–2.6.

(4) Basic topology of Euclidean space. Fundamental regions. GB 3.

(5) Symmetric groups. Cycle decomposition. Conjugacy classes. Generators.

(6) Reflections in abstract vector spaces. Roots. Root systems. Positive/negative roots. Coxeter groups. Simple roots. Simplicial cones. Fundamental regions for Coxeter groups. GB 4.1–4.2.

(7) Sylvester's criterion for the positive definiteness of a real symmetric matrix.

(8) Coxeter graphs. Dynkin diagrams. Irreducible and positive definition Coxeter graphs. Classification of Coxeter graphs. The crystallographic condition. Construction of Coxeter groups. Full classification of Coxeter groups. Orders of irreducible Coxeter groups. GB 5.1–5.4.

(9) Generators and relations for Coxeter groups. Length. Coxeter theorem. Coxeter element. GB 6.

(10) Applications to representation theory. The ubiquitousness of the ADE classification in mathematics and the physical sciences. Pictures of exceptional root systems.

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