University of Pennsylvania Department of Mathematics
Math 260 Honors Calculus II Spring Semester 2009
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Midterm exam 1, February 24, 2009 (solutions)

1. Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices and let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Let $T : V \to V$ be defined by $T(X) = AX$. Show that T is a linear transformation.
Solution. For any two matrices $X, Y \in V$ and scalar $a \in \mathbb{R}$ we have

$$\begin{array}{rcll} T(X+Y) & = & A(X+Y) & \text{definition of } T \\ & = & AX + AY & \text{distributivity of matrix multiplication} \\ & = & T(X) + T(Y) & \text{definition of } T \end{array}$$

and also

$$\begin{array}{rcll} T(aX) & = & A(aX) & \text{definition of } T \\ & = & a(AX) & \text{scalars commute with matrix multiplication} \\ & = & a\,T(X) & \text{definition of } T \end{array}$$

showing that T is linear.

Pick a basis of V and write the matrix $m(T)$ of T with respect to that basis (chosen for both the domain and codomain).
Solution. We'll choose the "standard" basis of V, namely

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We've already proven that this is a basis of the vector space of 2×2 matrices. Now calculate

$$\begin{array}{rcl} T(e_1) & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = e_1 \\ T(e_2) & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e_2 \\ T(e_3) & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = e_1 + e_3 \\ T(e_4) & = & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = e_2 + e_4 \end{array}$$

and put the coefficient vectors expressing these elements in terms of the chosen basis in the column of the matrix

$$m(T) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Let V_n be the vector space of all polynomials of degree $\leq n$. Consider the linear transformation $T : V_1 \to V_1$ defined by

$$T(p(x)) = x\,p'(x).$$

Compute the matrix of T with respect to the basis $1, x$ (chosen for both domain and codomain) and also the basis $1+x, 1+2x$ (chosen for both domain and codomain).

Solution. For the first basis, we calculate

$$T(1) = 0, \qquad T(x) = x$$

and then put the coefficient vectors representing these elements in terms of the basis $1, x$ in the columns of the matrix

$$m(T) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the second basis, we calculate

$$T(1+x) = x = -1(1+x) + (1+2x), \qquad T(1+2x) = 2x = -2(1+x) + 2(1+2x)$$

and then put the coefficient vectors representing these elements in terms of the basis $1, x$ in the columns of the matrix

$$m(T) = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}.$$

3. Let $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and find $\dim \ker(B)$ and $\dim \operatorname{im}(B)$.

Solution. First we'll check if B is injective, equivalently (since B is square), if B is surjective, equivalently, if $\det B \neq 0$. We calculate

$$\det B = 3 + 2 + 0 - 1 - 4 - 0 = 0,$$

and so we find that B is not injective, i.e. $0 < \dim \ker(B)$. But since B is not the zero matrix, it cannot have $\dim \ker(B) = 3$, so we're left with either $\dim \ker(B)$ equal 1 or 2. By the rank-nullity theorem, $\dim \operatorname{im}(B) = \operatorname{rank}(B)$ is also either 1 or 2.

The image of B is generated by the columns, and we note that the first two columns are linearly independent. Thus $\dim \operatorname{im}(B)$ is at least 2. Putting this together, we get that $\dim \operatorname{im}(B) = 2$ and $\dim \ker(B) = 1$.

An alternate solution involves Gauss-Jordan eliminating B to the reduced row-echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, from which we see that $\dim \ker(B) = 1$ and hence by the rank-nullity theorem, $\dim \operatorname{im}(B) = 2$.

4. Let $B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$ and find a basis of $\ker(B)$ and $\operatorname{im}(B)$.

Solution. First Gauss-Jordan eliminate B to the reduced-row echelon form $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ from which we see that $\operatorname{rank}(B) = 1$, hence any non-zero column will be a basis for the image (which is the columnspace), so pick the first column. From the reduced matrix, we find that $\ker(B)$ is the subspace

$$\ker(B) = \{(x, y, z) \in \mathbb{R}^3 : x - y + 2z = 0\}.$$

Any two linearly independent vectors, say $(1, 1, 0)$ and $(0, 2, 1)$ in the kernel, will be a basis of the kernel.

5. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$. Compute $\det(A)$, $\det(B)$, and $\det(AB)$.

Solution. Compute

$$\det(A) = 1 + 30 + 0 - 0 - 10 - 0 = 21.$$

We saw from problem 4 that $\det(B) = 0$. Then finally

$$\det(AB) = \det(A)\det(B) = 21 \cdot 0 = 0,$$

by the multiplicativity of the determinant.

6. Let $B = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, compute B^{10}.
Solution. We'll first try to diagonalize B. To this end we compute the eigenvalues of B as the roots of the characteristic polynomial,

$$\text{char}_B(t) = t^2 - \text{tr}(B)t + \det(B) = t^2 - 5t + 6 = (t-1)(t-6),$$

which are 1 and 6. Since the eigenvalues are distinct, we know that B is diagonalizable. To find a diagonalization, we find (a basis of) eigenvectors. Of course, we could solve the equation $(B - \lambda I)v = 0$ for each eigenvector λ. Here is a great trick for getting quickly at the eigenvectors of a 2×2 matrix (do this an an exercise, it's fun!):

if a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *has an eigenvalue* λ *then* $\begin{bmatrix} b \\ \lambda - a \end{bmatrix}$ *is a* λ*-eigenvector.*

Using this trick, we see that $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are eigenvectors for 1 and 6, respectively. As always, we can modify eigenvectors by scalars, so we might as well take $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Thus $P = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$ gives a diagonalization of B, i.e. $P^{-1}BP = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$. Now we can take powers,

$$\begin{aligned} B^{10} &= (P \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} P^{-1})^{10} = P \begin{bmatrix} 1 & 0 \\ 0 & 6^{10} \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6^{10} \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix} \frac{1}{-5} \\ &= \frac{1}{-5} \begin{bmatrix} -2 - 3 \cdot 6^{10} & -1 + 6^{10} \\ -6 + 6 \cdot 6^{10} & -3 - 2 \cdot 6^{10} \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2(1 + 9 \cdot 6^9) & 1 - 6^{10} \\ 6(1 - 6^{10}) & 3(1 + 4 \cdot 6^9) \end{bmatrix}. \end{aligned}$$

The powers of an integer matrix should be an integer matrix. Check for yourself that the entries of the above matrix are indeed divisible by 5.

7. Let A be a symmetric 5×5 matrix. Explain how you would compute A^{100}.
Solution. By a theorem from class, any symmetric matrix can be (orthogonally) diagonalized, i.e. we can find a matrix P so that $P^{-1}AP = D$ is a diagonal matrix. Then applying the conjugation power formula, we have that

$$A^{100} = (PDP^{-1})^{100} = PD^{100}P^{-1}$$

but the powers of a diagonal matrix are easy to compute, so we can compute this product of three matrices. The hard part is finding P.

8. True of false. Support your answers.

a) The rank of a matrix is equal to the number of its nonzero columns.
Solution. False. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has rank 1 but has 2 nonzero columns.

b) The $m \times n$ zero matrix is the only $m \times n$ matrix of rank 0.
Solution. True. Any nonzero entry in a column would contribute to the dimension of columnspace, hence to the rank.

c) Elementary row operation preserve rank.
Solution. True. Elementary row operation are equivalent to left multiplication by elementary matrices, which are all invertible. Multiplying by invertible matrices does not change the rank.

d) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.
Solution. True. The rank is equal to the dimension of the columnspace and also to the dimension of the rowspan. Since the rows span the rowspan (by definition), we can select from the rows a basis, i.e. a maximal linearly independent subset.

e) An $n \times n$ matrix is of rank at most n.
Solution. True. If there are n columns, then the maximal number of linearly independent columns is bounded by n.

f) An $n \times n$ matrix having rank n is invertible.
Solution. True. By the rank-nullity theorem, if an $n \times n$ matrix has rank n, then it has nullity 0, i.e. the kernel consists of the zero vector alone. This implies that the matrix is invertible.

9. Provide a reason why each of the following is *not* an inner product on the given real vector space.

a) $\langle (a,b), (c,d) \rangle = ac - bd$ on $\mathbb{R}^2$
Solution. Note that $\langle (0,1), (0,1) \rangle = -1$ contradicting the positivity condition of an inner product. Otherwise, this is bilinear, symmetric, and non-degenerate.

b) $\langle A, B \rangle = \mathrm{tr}(A + B)$ on $M_{2\times 2}(\mathbb{R})$
Solution. Letting $A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, we see that $\langle A, A \rangle = \mathrm{tr} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} = -2$ contradicting the positivity condition. This is not bilinear, but it is symmetric and non-degenerate.

c) $\langle f, g \rangle = \int_0^1 f'(t) g(t)\, dt$ on the vector space $P(\mathbb{R})$ of polynomials.
Solution. Note that $\langle 1, 1 \rangle = 0$ contradicting the non-degenerate condition. This is bilinear but neither positive nor symmetric.

10. Let $(V, \langle , \rangle)$ be any Euclidean vector space, i.e. a real vector space (not necessarily finite dimensional) with an inner product. Show that the composition of orthogonal transformations is orthogonal.

Solution. A transformation $T : V \to V$ is orthogonal if

$$\langle T(v), T(w) \rangle = \langle v, w \rangle, \qquad \text{for all } v, w \in V.$$

Let T and S be orthogonal transformations of V. Then for any $v, w \in V$, we have

$$\begin{aligned} \langle (T \circ S)(v), (T \circ S)(w) \rangle &= \langle T(S(v)), T(S(w)) \rangle && \text{by definition of composition} \\ &= \langle S(v), S(w) \rangle && \text{since } T \text{ is orthogonal} \\ &= \langle v, w \rangle && \text{since } S \text{ is orthogonal} \end{aligned}$$

and so the composition $T \circ S$ is orthogonal as well.