University of Pennsylvania Department of Mathematics
Math 260 Honors Calculus II Spring Semester 2009
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Midterm exam 2, April 7, 2009 (solutions)

1. Write a basis for the space of pairs (u, v) of smooth functions $u, v : \mathbb{R} \to \mathbb{R}$ that satisfy the system of linear differential equations

$$\begin{aligned} u' &= -2u + 2v \\ v' &= 2u + v \end{aligned}$$

Solution. The system can be expressed as a matrix equation

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

From class, we know a basis of solutions of this equation if we can diagonalize the matrix. To this end, we find the eigenvalues as the roots of the characteristic polynomial

$$\mathrm{char}(t) = t^2 + x - 6 = (x + 3)(x - 2),$$

which are -3 and 2. Since the eigenvalues are distinct, we know that the matrix is diagonalizable. To find a diagonalization, we find (a basis of) eigenvectors. Of course, we could now solve the eigenvector condition equation for each eigenvalue λ. Here is a great trick for quickly getting at the eigenvectors of a 2×2 matrix (prove this as an exercise, it's fun!):

if a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *has an eigenvalue* λ *then* $\begin{bmatrix} b \\ \lambda - a \end{bmatrix}$ *is a* λ*-eigenvector.*

Using this trick, we see that $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are eigenvectors for -3 and 2, respectively. As always, we can modify eigenvectors by scalars, so we might as well take $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Now we can use the theorem from class that a basis of solution is given by

$$e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

2. Determine whether the following sets in the plane are open closed or neither. Also describe the boundary set.

a) $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$

Solution. Open. This set is the intersection of the open ball of radius 2 with the complement of the closed ball of radius 1. So it's the intersection of (two) open sets, so is open. The boundary is the disjoint union of the circles of radius 1 and 2, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ or } 2\}$.

b) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Solution. Closed. This set is the intersection of the closed ball of radius 1 with complement of the open ball of radius 1. So it's the intersection of closed sets, so is closed. It also can be seen to contain all of it's limit points. It's its own boundary.

c) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

Solution. Closed. This set is the closed ball of radius 1. It's boundary is the circle of radius 1.

d) $\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 < 2\}$

Solution. Neither. It's not open because while $(1, 0)$ is in the set, no ball around $(1, 0)$ is contained in the set. It's not closed because $(2, 0)$ is seem to be a limit point of the set, but is not contained in it. The boundary is the disjoint union of the circles of radius 1 and 2, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ or } 2\}$.

e) $\mathbb{R}^2 \setminus \{$ nonnegative $x-$axis $\}$

Solution. Open. The nonnegative $x-$axis is closed (in $\mathbb{R}^2$) because it's seen to contain all of its limit points. Thus it's complement is open. The boundary is the nonnegative $x-$axis.

f) $\mathbb{R}^2 \setminus \{$ positive $x-$axis $\}$

Solution. Neither. The positive $x-$axis is not open (in $\mathbb{R}^2$) because *no* ball containing a point on the axis is strictly contained in the axis. The positive $x-$axis is not closed (in $\mathbb{R}^2$) because the origin is seen to be a boundary point but is not contained in the positive $x-$axis. Since the positive $x-$axis is neither open nor closed, then neither is its complement. The boundary is the nonnegative $x-$axis.

g) $\mathbb{R}^2 \setminus \{$ integers $\}$

Solution. Open. The integers are closed (in $\mathbb{R}^2$) since they contain all their boundary points. Indeed, any convergent sequence of integers is eventually constant. Thus their complement is open. The boundary is the set of integers.

3. Determine if each of the following limits exists. If so, give the limit and an explanation, if not, prove it.

a) $\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$

Solution. Does not exist. Along the line $y = 0$, we have that

$$\lim_{(x,0)\to(0,0)} \left(\frac{x^2}{x^2}\right)^2 = \lim_{(x,0)\to(0,0)} 1 = 1$$

while along the line $x = y$, we have

$$\lim_{(x,x)\to(0,0)} \left(\frac{x^2 - x^2}{x^2 + x^2}\right)^2 \lim_{(x,x)\to(0,0)} \frac{0}{2} = 0$$

and so we see that the limit cannot exist.

b) $\lim_{(x,y)\to(1,1)} \frac{xy^2}{x^2 + y^2}$

Solution. The function $\frac{xy^2}{x^2+y^2}$ is a quotient of continuous functions (polynomials) and the denominator does not vanish at $(1, 1)$. Thus by rules for continuity, the quotient is continuous $(1, 1)$, hence

$$\lim_{(x,y)\to(1,1)} \frac{xy^2}{x^2 + y^2} = \left.\frac{xy^2}{x^2 + y^2}\right|_{(1,1)} = \frac{1}{2}.$$

4. Find the equation of the tangent plane to the graph of $f(x, y) = 9 - x^2 - y^2$ at the point $(1, -2)$.

Solution. The graph of f in $\mathbb{R}^3$ can be realized as the level surface (or hypersurface)

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}, \quad g(x, y, z) = x^2 + y^2 + z - 9.$$

Our point is then $(1, -2, 4)$ on the level surface.

> *The general formula for the tangent plane T_pS to a point p on any level surface $S = \{x \in \mathbb{R}^n : g(x) = 0\}$ is given by*
>
> $$T_pS = \{v \in \mathbb{R}^n : D_pg(v - p) = 0\},$$
>
> *as long as D_pg is* **not** *the zero map!*

In our situation and in standard coordinates, we have

$$D_{(1,-2,4)}g = \left[\frac{\partial g}{\partial x}\bigg|_{(1,-2,4)}, \frac{\partial g}{\partial y}\bigg|_{(1,-2,4)}, \frac{\partial z}{\partial x}\bigg|_{(1,-2,4)} \right] = [2, -4, 1].$$

Thus the required tangent plane is

$$T_{(1,-2,4)}S = \{(x, y, z) \in \mathbb{R}^3 : 2(x - 1) - 4(y + 2) + (z - 4) = 0\} = \{(x, y, z) \in \mathbb{R}^3 : 2x - 4y + z = 14\}.$$

5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$. For each of the following, use examples and statements from class and/or the homework to determine whether such a function exists.

a) f is differentiable at a but no continuous at a.

Solution. Does not exist. From a theorem from class, if f is differentiable at a point, then it is continuous there.

b) All the partial derivatives exist at a but f is not differentiable at a.

Solution. On homework 9, there was an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

whose partial derivatives exist at $(0, 0)$ but is not differentiable there.

c) The equality $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$.

Solution. The wording of this part is unintentionally ambiguous. Strictly speaking, yes, any function whose second partial derivatives exist and are continuous everywhere has an equality of mixed partials. For example, the zero function (or any polynomial) satisfies this criterion. The question was supposed to be, "Does there exist a function where the mixed partials disagree?" Yes, there exist such functions. Examples of such are outlined in Apostol's book.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (9 - x^2 - y^2, e^{xy})$. Explain why f is differentiable at every point $(a, b) \in \mathbb{R}^2$. Compute the total derivative $D_{(a,b)}f$.

Solution. A theorem from class says that $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable if and only if the component functions f_i of f are differentiable. Here, $9 - x^2 - y^2$ is a polynomial so is differentiable everywhere, and also e^{xy} is differentiable everywhere. Finally, we compute, using the Jacobian matrix expressed in the standard basis,

$$D_{(a,b)}f = \begin{bmatrix} \frac{\partial f_1}{\partial x}\Big|_{(a,b)} & \frac{\partial f_1}{\partial y}\Big|_{(a,b)} \\ \frac{\partial f_2}{\partial x}\Big|_{(a,b)} & \frac{\partial f_2}{\partial y}\Big|_{(a,b)} \end{bmatrix} = \begin{bmatrix} -2a & -2b \\ e^{a+b} & e^{a+b} \end{bmatrix}.$$

for all $(a, b) \in \mathbb{R}^2$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = \sin^2(x + y), \qquad g(u, v) = (u^2 + uv^3, uv).$$

Compute $D_{(a,b)}(f \circ g)$ for any $(a, b) \in \mathbb{R}^2$.

Solution. We'll use the chain rule

$$D_{(a,b)}(f \circ g) = D_{g(a,b)} f \circ D_{(a,b)} g.$$

To this end, using the Jacobian matrix with respect to the standard bases, compute

$$D_{(x,y)} f = [2\sin(x + y)\cos(x + y), 2\sin(x + y)\cos(x + y)] = 2\sin(x + y)\cos(x + y)\,[1, 1],$$

and

$$D_{(u,v)} g = \begin{bmatrix} 2u + v^3 & 3uv^2 \\ v & u \end{bmatrix}.$$

Then

$$\begin{aligned} D_{(a,b)}(f \circ g) &= 2\sin(x + y)\cos(x + y)|_{(a^2+ab^3,ab)}\,[1, 1] \begin{bmatrix} 2a + b^3 & 3ab^2 \\ b & a \end{bmatrix} \\ &= 2\sin(a^2 + ab^3 + ab)\cos(a^2 + ab^3 + ab)\,[2a + b + b^2, a + 3ab^2]. \end{aligned}$$

8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = xy + e^{xy} + z^2$. Determine the critical point(s) of f and at each one, write the quadratic approximation. What can you say about the extreme values (local/global min/max) of f?

Solution. First note that f is differentiable on $\mathbb{R}^3$. Now, using the Jacobian matrix with respect to the standard basis, compute

$$D_{(x,y,z)} f = [y + ye^{xy}, x + xe^{xy}, 2z].$$

Then $(x, y, z) \in \mathbb{R}^3$ is a critical point of f if $D_{(x,y,z)} = (0, 0, 0)$. This is equivalent to $z = 0$, $x(1 + e^{xy}) = 0$ and $y(1 + e^{xy}) = 0$. But now since $1 + e^{xy} > 0$, we finally get that (x, y, z) is a critical point for f if and only if $(x, y, z) = (0, 0, 0)$.

For the quadratic approximation, we'll need the Hessian matrix of f at the critical point,

$$H_{(0,0,0)} f = \left.\begin{bmatrix} y^2e^{xy} & 1 + e^{xy} + xye^{xy} & 0 \\ 1 + e^{xy} + xye^{xy} & x^2e^{xy} & 0 \\ 0 & 0 & 2 \end{bmatrix}\right|_{(0,0,0)} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The quadratic approximation of f at $(0, 0, 0)$ is now

$$\begin{aligned} f(u, v, w) &\approx f(0, 0, 0) + D_{(0,0,0)} f(u, v, w) + \frac{1}{2}(u, v, w) H_{(0,0,0)} f(u, v, w)^t \\ &= 1 + 0 + \frac{1}{2}(4uv + 2w^2) = 1 + 2xy + z^2. \end{aligned}$$

Note that we could have guessed this from the beginning, since the quadratic term in the quadratic approximation of e^{xy} is just xy (coming from the Taylor expansion), while xy and z^2 are their own quadratic approximations.

Finally, note that $\det H_{(0,0,0)} f = -8$ and has at least one eigenvalue of 2 (the vector $(1, 1, 1)$ works). Thus the product of the remaining two eigenvalues is -4, hence one of them is negative. Thus the Hessian is indefinite at $(0, 0, 0)$, indicating that f has a saddle point there. As for global min or max, the xy term in f guarantees that f gets arbitrarily large in both positive and negative directions. So there are no local and no global mins or maxs.

9. Using the (limit) definition of the total derivative, show that $[1, 1]$ is the total derivative of the function $f(x, y) = x + y$ for any point in $\mathbb{R}^2$.

Solution. For any $(x,y) \in \mathbb{R}^2$, compute

$$\begin{aligned} \lim_{(u,v)\to(0,0)} \frac{|f(x+u,y+v) - f(x,y) - [1,1](u,v)|}{|(u,v)|} &= \lim_{(u,v)\to(0,0)} \frac{|x+u+y+v-(x+y)-(u+v)|}{\sqrt{u^2+v^2}} \\ &= \lim_{(u,v)\to(0,0)} \frac{0}{\sqrt{u^2+v^2}} = 0 \end{aligned}$$

and hence the linear function $[1,1]$ (i.e. dotting a vector with $[1,1]$) is indeed the derivative of f at (x,y).

10. EXTRA CREDIT Write a basis for the space of pairs (u,v) of smooth functions $u,v:\mathbb{R}\to\mathbb{R}$ that satisfy the system of linear differential equations

$$\begin{aligned} u' &= 9u + 10v \\ v' &= -10u - 11v \end{aligned}$$

Solution. The system can be expressed as a matrix equation

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ -10 & -11 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

From class, we know a basis of solutions of this equation if we can diagonalize the matrix. To this end, we find the eigenvalues as the roots of the characteristic polynomial

$$\mathrm{char}(t) = t^2 + 2x + 1 = (x+1)^2,$$

and now we realize that afterall, the matrix is not diagonalizable. A theorem from class says that the columns of the matrix exponential will be a basis of solutions of the system. To this end, note that we can decompose

$$\begin{bmatrix} 9 & 10 \\ -10 & -11 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 10 & 10 \\ -10 & -10 \end{bmatrix}$$

into commuting matrices (since scalar matrices commute with everything). First, recall that

$$e^{t\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Also, note that using the Taylor expansion, we have

$$e^{t\begin{bmatrix} 10 & 10 \\ -10 & -10 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 10t & 10t \\ -10t & -10t \end{bmatrix} + \text{zero matrices}$$

since all higher powers of the matrix vanish. Now using the product formula for the exponential, we have

$$\begin{aligned} e^{t\begin{bmatrix} 9 & 10 \\ -10 & -11 \end{bmatrix}} &= e^{t\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}} e^{t\begin{bmatrix} 10 & 10 \\ -10 & -10 \end{bmatrix}} \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1+10t & 10t \\ -10t & 1-10t \end{bmatrix} \\ &= \begin{bmatrix} (1+10t)e^{-t} & 10te^{-t} \\ -10te^{-t} & (1-10t)e^{-t} \end{bmatrix} \end{aligned}$$

and so the columns of this matrix form a basis for the system of differential equations. One can further simplify this basis to

$$\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{bmatrix} (1+10t)e^{-t} \\ -10te^{-t} \end{bmatrix}.$$