University of Pennsylvania Department of Mathematics
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Practice final exam solutions

1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + y, x - y)$.

a) If F denotes a force field, then show that the work done by this field to move a particle along the curve $\alpha : [a, b] \to \mathbb{R}^2$ defined by $\alpha(t) = (f(t), g(t)$, only depends on the values $f(a), f(b), g(a), g(b)$.

Solution. The work done is calculated by the line integral

$$\begin{aligned}
\int F \cdot d\alpha &= \int_a^b F(\alpha(t)) \cdot \alpha'(t)\, dt \\
&= \int_a^b (f(t) + g(t), f(t) - g(t)) \cdot (f'(t), g'(t))\, dt \\
&= \int_a^b \big(f(t)f'(t) + g(t)f'(t) + f(t)g'(t) - g(t)g'(t)\big)dt \\
&= \int_{f(a)}^{f(b)} u\, du + \left(g(t)f(t)|_a^b - \int_a^b g'(t)f(t)\, dt\right) + \int_a^b f(t)g'(t)\, dt - \int_{g(a)}^{g(b)} v\, dv \\
&= \frac{1}{2}(f(b)^2 - f(a)^2) + f(b)g(b) - f(a)g(a) - \frac{1}{2}(g(b)^2 - g(a)^2) \\
&= \frac{1}{2}((f(b)^2 + 2f(b)g(b) - g(b)^2) - (f(a)^2 + 2f(a)g(a) - g(a)^2))
\end{aligned}$$

using integration by parts and the substitutions $u = f(t)$ and $v = g(t)$.

This answer might lead one to guess that there's a *potential function*, i.e. that $F = \nabla\phi$, for some $\phi : \mathbb{R}^2 \to \mathbb{R}$. Indeed, letting $\phi(x, y) = \frac{1}{2}(x^2 + 2xy - y^2)$ works!

b) Find the amount of work done when $f(a) = 1$, $f(b) = 2$, $g(a) = 3$, $g(b) = 4$.

Solution. The amount of work done is

$$\frac{1}{2}((2^2 + 2 \cdot 2 \cdot 4 - 4^2)^2 - (1^2 + 2 \cdot 1 \cdot 3 - 3^2)) = 3.$$

2. A sphere is inscribed in a cylinder. The sphere is sliced by two parallel planes that are perpendicular to the axis of the cylinder. Show that the portions of the sphere and cylinder lying between these planes has the same surface area.

Solution. We might as well assume that the sphere is centered at the origin, has radius a, and that the cylinder (with radius a) has axis along the z-axis. Now the sphere has parameterization $r : [0, 2\pi] \times [-\pi/2, \pi/2] \to \mathbb{R}^3$ given by

$$r(\theta, \varphi) = (a\cos(\theta)\cos(\varphi), a\sin(\theta)\cos(\varphi), a\sin(\varphi))$$

Let's say that the planes are $z = b$ and $z = c$ where $-a \leq b \leq c \leq a$. Then The portion S of the sphere between the two planes is parameterized (via r) by the subset $[0, 2\pi] \times$

$[\sin^{-1}(b/a), \sin^{-1}(c/a)]$. Using the parameterization formula for surface integrals, we have

$$\begin{aligned}\int_S 1 &= \int_{\theta=0}^{2\pi}\int_{\varphi=\sin^{-1}(b/a)}^{\sin^{-1}(c/a)} \sqrt{\det\big((D_{(\theta,\varphi)}r)^t D_{(\theta,\varphi)}r\big)}\, d\varphi d\theta \\ &= \int_{\theta=0}^{2\pi}\int_{\varphi=\sin^{-1}(b/a)}^{\sin^{-1}(c/a)} a^2|\cos(\varphi)|\, d\varphi d\theta.\end{aligned}$$

The area of the portion C of the cylinder between the two planes is

$$2\pi a(c-b).$$

Check for yourself that these agree.

3. Let $D \subset \mathbb{R}^2$ be the solid diamond-shaped region in the plane bounded by the points $(0,0)$, $(1,1)$, $(2,0)$, $(1,-1)$. Compute $\int_D (x^2+y^2)$ using the change of coordinates $(u,v) = (x-y, x+y)$.

Solution. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2+y^2$ and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(x,y) = (x-y, x+y)$. To use the change of variables formula (integral parameterization formula), we want to find a nice subset $S \subset \mathbb{R}^2$ so that $\Phi(R) = D$, i.e. $R = \Phi^{-1}(D)$. Note that Φ is a linear map given by the matrix $\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$, so the function theoretic inverse Φ^{-1} is also a linear map given by the inverse matrix $\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$. Since Φ and Φ^{-1} are linear maps, they take parallelograms to parallelograms. Then $R = \Phi^{-1}(D)$ is the solid region in the plane bounded by the point $(0,0) = \Phi^{-1}(0,0)$, $(1,0) = \Phi(1,1)$, $(1,-1) = \Phi(2,0)$, $(0,-1) = \Phi(1,-1)$, i.e. $R = [0,1] \times [-1,0]$ is the square in the fourth quadrant of side length 1 with one corner at the origin. So $\Phi : R \to D$ is a parameterization of D by a nice region R in which we can use a straightforward iterated Fubini integral.

Now we'll consider the parameterizaion factors in the integral. We have

$$(f\circ\Phi)(x,y) = (x-y)^2 + (x+y)^2 = 2(x^2+y^2).$$

Also

$$D_{(x,y)}\Phi = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

since Φ is linear (so equals its own total derivative) and so

$$\sqrt{\det\big((D_{(x,y)}\Phi)^t D_{(x,y)}\Phi\big)} = |\det D_{(x,y)}\Phi| = 2.$$

Finall we can use the parameterization formula

$$\begin{aligned}\int_D f &= \int_R (f\circ\Phi)\sqrt{\det\big((D\Phi)^t D\Phi\big)} \\ &= 4\int_{x=0}^{1}\int_{y=-1}^{0} (x^2+y^2)\, dydx \\ &= 4\int_{x=0}^{1} (x^2+\frac{1}{3})\, dx = 4\cdot\frac{2}{3} = \frac{8}{3}.\end{aligned}$$

By the way, if you wanted to compute the original integral with an iterated integral, it would be

$$\int_{x=0}^{1}\int_{y=-x}^{x} (x^2+y^2)\, dydx + \int_{x=1}^{2}\int_{y=-2+x}^{2-x} (x^2+y^2)\, dydx$$

which you can check for yourself gives the same answer, albeit after a long ugly calculation. Try it!

4. Find the volume of the solid tetrahedral region bounded in the first octant by the plane $3x + 4y + 2z = 12$. Set up the complete iterated integral.

Solution. The solid region we are considering is

$$S = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, 3x + 4y + 2z \leq 12\}.$$

First note that the intersection of the plane with the positive x-, y-, and z-axes is 4, 3, and 6, respectively. In particular, x has the freedom to go from 0 to 4 in S. Next, we'll consider the freedom of y for each fixed x. For this, projecting the region to the x-y-plane (i.e. setting $z = 0$), we get the equation $3x + 4y = 12$. Solving for y in terms of x gives $y = 3 - \frac{3}{4}x$. This shows that for a fixed x, y has the freedom to move from 0 to $3 - \frac{3}{4}x$. Finally, we consider the freedom of z for each fixed x and y. Solving the equation of the plane for z in terms of x and y yields $z = 6 - \frac{3}{2}x - 2y$. This shows that for a fixed x and y, z has the freedom to move from 0 to $z = 6 - \frac{3}{2}x - 2y$ in the region. Thus the iterated integral is

$$\int_{x=0}^{4} \int_{y=0}^{3-\frac{3}{4}x} \int_{z=0}^{6-\frac{3}{2}x-2y} 1 \, dz dy dx.$$

5. Using Green's theorem, evaluate the line integral $\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F}(x, y) = (x^3 - 2y^3, x^3 + 2y^3)$ and $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2, x \geq 0, y \geq 0\}$, for some fixed positive real number a.

Solution. Since the region D is contained in the x-y-plane, we can use version (1) of Green's theorem. In this case,

$$\begin{aligned} \int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{\partial D} (x^3 - 2y^3)\, dx + (x^3 + 2y^3)\, dy \\ &= \int_D \left(\frac{\partial(x^3 + 2y^3)}{\partial x} - \frac{\partial(x^3 - 2y^3)}{\partial y} \right) \\ &= \int_D (3x^2 + 6y^2). \end{aligned}$$

Now to compute this integral, we use the change of coordinates (parameterization) formula to use circular coordinates. Indeed, let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\Phi(r, \theta) = (r\cos(\theta), r\sin(\theta))$. Then $\Phi : [0, a] \times [0, \pi/2] \to D$ gives a parameterization of the region D in terms of a nice box. As usual with circular coordinates, the parameterization factor is given by

$$\sqrt{\det\big((D_{(r,\theta)}\Phi)^t D_{(r,\theta)}\Phi\big)} = |\det D_{(r,\theta)}\Phi| = \left| \det \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \right| = |r|.$$

Thus we use the parameterization formula

$$\begin{aligned} \int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{r} &= 3 \int_D (x^2 + 2y^2) \\ &= 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} (r^2\cos^2(\theta) + 2r^2\sin^2(\theta))|r| \, d\theta dr \\ &= 3 \int_{r=0}^{a} r^3 \int_{\theta=0}^{\pi/2} (1 + \sin^2(\theta))\, d\theta \, dr \\ &= 3 \int_{r=0}^{a} r^3 \left(\frac{\pi}{2} + \frac{\pi}{4} \right) dr = \frac{9\pi a^4}{16}. \end{aligned}$$

Do you think this is easier than computing the original line integral? First, there are three pieces to the curve ∂D,

$$\gamma_1(t) = (a\cos(t), a\sin(t)), 0 \leq t \leq \pi/2, \quad \gamma_2(t) = (0, t), a \geq t \geq 0, \quad \gamma_3(t) = (t, 0), 0 \leq t \leq a,$$

so the line integral breaks into three integrals

$$\begin{aligned}
\int_{\partial D} \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{\pi/2} (a^3\cos^3(t) - 2a^3\sin^3(t), a^3\cos^3(t) + 2a^3\sin^3(t))\cdot\gamma_1'(t)\,dt \\
&\quad + \int_a^0 (-2t^3, 2t^3)\cdot\gamma_2'(t)\,dt + \int_0^a (t^3, t^3)\cdot\gamma_3'(t)\,dt \\
&= a^4\int_0^{\pi/2} (-\cos^3(t)\sin(t) + 2\sin^4(t) + \cos^4(t) + 2\sin^3(t)\cos(t))\,dt - 2\int_0^a t^3\,dt + \int_0^a t^3\,dt.
\end{aligned}$$

Then you will need to do freshman calculus tricks to compute this. What do you think is easier?

6. Using Stokes' Theorem, compute the surface integral $\int_{\partial S} \boldsymbol{F}\cdot\boldsymbol{n}$ where $\boldsymbol{F} = (x^3z, -xz^2, 3)$ and where S is the solid region $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 4z^2, 0\le z\le 1\}$ and $\boldsymbol{n}$ is some choice of unit normal. NOTE: there's a change of notation from the original statement of the problem.

Solution. By Stokes' Theorem we have

$$\begin{aligned}
\int_{\partial S} \boldsymbol{F}\cdot\boldsymbol{n} &= \int_S \mathrm{div}\boldsymbol{F} \\
&= \int_{z=0}^1 \int_{y=-2z}^{2z} \int_{x=-\sqrt{4z^2-y^2}}^{\sqrt{4z^2-y^2}} 3x^2z\,dxdydz
\end{aligned}$$

constructing the iterated Fubini integral just as in problem 4. But this integral looks hard, but doable with appropriate trig substitutions.

Instead, let's change to cylindrical coordinates. Define $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$. We see that the region

$$R = \{(r,\theta,z)\in\mathbb{R}^3 : 0\le z\le 1, 0\le\theta\le 2\pi, 0\le r\le 2z\}$$

parameterizes S via Φ, i.e. $\Phi : R\to S$. The parameterization factor is

$$|\det D_{(r,\theta,z)}\Phi| = \left|\det\begin{bmatrix}\cos(\theta) & -r\sin(\theta) & 0\\ \sin(\theta) & r\cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}\right| = |r|.$$

Finally, using the parameterization formula, we have

$$\begin{aligned}
\int_{\partial S} \boldsymbol{F}\cdot\boldsymbol{n} &= \int_S \mathrm{div}\boldsymbol{F} \\
&= \int_R (\mathrm{div}\boldsymbol{F}\circ\Phi)\,|r| \\
&= 3\int_{\theta=0}^{2\pi}\int_{z=0}^1\int_{r=0}^{2z} r^2\cos^2(\theta)z|r|\,drdzd\theta \\
&= 3\int_{\theta=0}^{2\pi}\cos^2(\theta)\int_{z=0}^1 z\left[\frac{r^4}{4}\right]_0^{2z} dzd\theta \\
&= 12\int_{\theta=0}^{2\pi}\cos^2(\theta)\int_{z=0}^1 z^5\,dzd\theta \\
&= 2\int_{\theta=0}^{2\pi}\cos^2(\theta)\,d\theta = 2\pi.
\end{aligned}$$

7. Let $S = \mathbb{R}^2 \setminus \{(0,0)\}$ and define $\boldsymbol{F} : S \to \mathbb{R}^2$ by

$$\boldsymbol{F}(x,y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right).$$

Compute the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for any closed curve $C \subset \mathbb{R}^2$ not enclosing the origin.

Solution. Since the problem is asking to compute a line integral for a pretty much arbitrary curve, the answer probably doesn't depend on the curve. So let's just say we have any curve C no enclosing the origin and let $R \subset \mathbb{R}^2$ be the region it bounds. Then by Green's theorem version (1) (which we can apply, since $\boldsymbol{F}$ is nice in the region R),

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{\partial R} \frac{y}{x^2+y^2}\, dx + \frac{-x}{x^2+y^2}\, dy \\ &= \int_R \left(\frac{\partial}{\partial x} \frac{-x}{x^2+y^2} - \frac{\partial}{\partial y} \frac{y}{x^2+y^2} \right) \\ &= \int_R 0 = 0. \end{aligned}$$

8. Find a basis for the vector space of 3×3 symmetric matrices. Here symmetric means $A^t = A$.

Solution. A basis is given by e_{ij} for all $1 \le i \le j \le 3$, where e_{ij} is the 3×3 matrix of all zeros except for a 1 in the ij and ji spots. If $i = j$ then the matrix e_{ii} just has a 1 in the ii spot. For example

$$e_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In total, there are 6 such matrices, so the vector space of 3×3 symmetric matrices has dimension 6.

9. Let V_n be the vector space of polynomials of degree $\le n$. Consider the linear transformation $T : V_1 \to V_3$ defined by

$$T(p(x) = x^2 \cdot p(x-1).$$

Compute a basis of $\ker(T)$ and $\operatorname{im}(T)$.

Solution. Choose bases $1, x$ and $1, x, x^2, x^3$ for V_1 and V_3, respectively. Then

$$\begin{aligned} T(1) &= x^2 \\ T(x) &= x^2(x-1) = x^3 - x^2 \end{aligned}$$

so the matrix of T with respect to these bases is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

In particular, we see that the rank of T is 2, so by the rank-nullity theorem, the kernel of T has dimenion 0, i.e. $\ker(T) = \{0\}$ and T is injective. The image of T is spanned by x^2 and $x^3 - x^2$, which are linearly independent, so form a basis. A nicer basis is x^2 and x^3, which is gotten by taking linear combinations.

10. Let $A = \begin{bmatrix} a & b & c \\ b & a & c \end{bmatrix}$ where $a \neq \pm b$. Compute the dimensions of $\ker(A)$ and $\operatorname{im}(A)$.

Solution. The condition $a \neq \pm b$ implies, in particular, that a and b are both not zero. So A is never the zero matrix, so can never have rank 0. Since A is a 2×3 matrix, $\dim\ker(A) \leq 3$, $\dim\operatorname{im}(A) \leq 2$, and $\dim\ker(A) + \dim\operatorname{im}(A) = 3$. Let's consider the image, which is the column span of A. We already see that the vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} b \\ a \end{bmatrix}$ are linearly independent. Indeed,

$$\det \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a^2 - b^2,$$

which is never zero as long as $a \neq \pm b$. In particular, the image of A has dimension at least 2, thus is must have dimension 2. By the rank-nullity theorem, the dimension of the kernel is 1. By the way, a basis for the kernel is given by the (column) vector $(\frac{c}{a+b}, \frac{c}{a+b}, 1)^t$.

11. Let A be an $n \times n$ real matrix.

a) What are the possible values of $\det A$ if A is symmetric.

Solution. Any real number. Indeed, for any real number a, the $n \times n$ diagonal matrix with diagonal consisting of one a and $n-1$ 1's (and all the off-diagonal entries zero) is symmetric and has determinant a.

b) What are the possible values of $\det A$ if A is invertible.

Solution. Any non-zero real number. We know that an $n \times n$ invertible matrix has non-zero determinant. Now we'll show that the determinant achieves all non-zero values. Given any non-zero real number a, the diagonal matrix considered above is invertible and has determinant a.

12. For all $(x, y) \neq (0, 0)$, define

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}.$$

Can $f(0, 0)$ be assigned to make f continuous at the origin?

Solution. It's well-known from calculus (when you find the derivative of sin) that

$$\sin'(0) = \lim_{z\to 0} \frac{\sin(z)}{z} = \cos(0) = 1,$$

and that the function

$$g(z) = \begin{cases} \frac{\sin(z)}{z} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

is continuous. Thus we should set $f(0, 0) = 1$.

To prove that f, as defined, is continuous, let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a continuous curve in the plane, with $\gamma(0) = (0, 0)$. Note that $r : \mathbb{R}^2 \to \mathbb{R}$ defined by $r(x, y) = x^2 + y^2$ is continuous. Finally, we now realize f as a composition of functions $f = g \circ r$, which are each continuous everywhere, making f continuous everywhere, with $f(0, 0) = g(r(0, 0)) = g(0) = 1$.

15. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \sqrt{|xy|}$, is not differentiable at the origin $(0, 0)$.

Solution. A good necessary criterion for a function f to be differentiable at p is that the *difference quotient*,

$$\frac{|f(p + h) - f(p)|}{|h|},$$

is bounded as $h \to p$. In our situation, unfortunately our difference quotient is

$$\frac{|f(u,v) - f(0,0)|}{|(u,v)|} = \frac{\sqrt{|uv|}}{\sqrt{u^2+v^2}},$$

for $(u,v) \to (0,0)$, which is bounded (by $\frac{1}{2}$). So we must use a different strategy.

We want to ask if a linear map $D_{(0,0)}f$ exists so that the limit

$$\lim_{h \to 0} \frac{|f(h) + D_{(0,0)}f(h)|}{|h|} = 0.$$

Now along the path $h = (a,b)$ with $a = b$, we have

$$\lim_{\substack{(a,b)\to(0,0)\\ a=b}} \frac{||a| + aD_{(0,0)}f(1,1)|}{|a|} = \lim_{a\to 0} \left|1 + \frac{a}{|a|} D_{(0,0)}(1,1)\right|,$$

which gives a limit of either $|1+D_{(0,0)}f(1,1)|$ or $|1-D_{(0,0)}f(1,1)|$ depending on whether the pather is approaching from the first or third quadrant in the plane. Now both of these values can't be 0, so the original limit can't equal 0. Thus f is not differentiable at the origin.