UNIVERSITY OF PENNSYLVANIA DEPARTMENT OF MATHEMATICS
Math 260 Honors Calculus II Spring Semester 2009
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Selected solutions
HW 1, due January 22, 2009
Exercises from Apostol Vol. II
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2.4 #25 Let $V = C^\infty(-1,1)$ be the real vector space of infinitely differentiable real-valued (also called *smooth*) functions $f : (-1,1) \to \mathbb{R}$. Let $T : V \to V$ be the map defined by

$$T(f)(x) = x\,f'(x), \quad \text{for all } x \in (-1,1).$$

Note that $T(f) \in V$ since the derivatives and products of smooth functions are again smooth. This is a **general warning**: the derivative of a (once-)differentiable function may not be differentiable. An example to keep in mind should be the "signed parabola" $q : \mathbb{R} \to \mathbb{R}$ defined by

$$q(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

which is everywhere differentiable and whose derivative is the absolute value, $q'(x) = |x|$ which is not differentiable at $x = 0$. Unless he explicitly states otherwise, anytime Apostol says "differentiable" he really means "smooth". By the way, while Apostol uses "null space" and "range", I'll employ the more modern usage "kernel" and "image" and I'll write $\ker(T)$ and $\mathrm{im}(T)$, respectively. As always, "vector space" will mean here "real vector space".

Rant. Before any solutions, I'd like to address a grammatical pet peeve of mine concerning arguments of functions/maps. If $f : (-1,1) \to \mathbb{R}$ is a function, then it takes an argument, say x, which is a real number in the interval $(-1,1)$, and produces an value $f(x)$, which is a real number. Now the map $T : V \to V$ takes an argument, say f which is a smooth function $f : (-1,1) \to \mathbb{R}$, and produces a value $T(f)$ which is another smooth function $T(f) : (-1,1) \to \mathbb{R}$. Now writing

$$T(f) = x\,f'(x)$$

is **incorrect**, while writing

$$T(f)(x) = x\,f'(x)$$

is **correct** since $T(f)$ is a function and so takes an argument which is a real number (above it was x). It's all about getting the arguments right. Now here's a situation you'll encounter often. You want to show that two functions, say f and g, are equal:

> two functions f and g are equal (and write $f = g$) if and only if they have the same domain and codomain $f : X \to Y$ and $g : X \to Y$, and for every element $x \in X$, the equality $f(x) = g(x)$ is satisfied in Y.

Pretty much the only way to show two functions are equal is to test them on elements in the domain. Be explicit about this. For example, if you want to show that two linear transformations, say $DT - TD : V \to V$ and $D : V \to V$ are equal, then for every $f \in V$, you need to verify that the equality $(DT - TD)(f) = D(f)$ of functions holds, which in turn requires you to test on real numbers $x \in (-1, 1)$. This might proceed as follows,

$$\begin{aligned} (DT - TD)(f)(x) &= DT(f)(x) - TD(f)(x) \\ &= \frac{d}{dx}(x\, f'(x)) - x\, f''(x) \\ &= x\, f''(x) + f'(x) - x\, f''(x) = f'(x) \\ &= D(f)(x) \end{aligned}$$

which proves that $(DT - TD)(f) = D(f)$ are equal as functions for all $f \in V$, which in turn proves that $DT - TD = D$ as linear transformations from V to V. Get it?

On with the solution. The problem asked to determine if T is linear and if so, to "describe" it's kernel and image. We can actually give quite a good description.

Proposition 1. *The map $T : V \to V$ is a linear transformation. The kernel of T consists of the one-dimensional vector space*

$$\ker(T) = \{f \in V \,:\, \exists c \in \mathbb{R}, f(x) = c, \forall x \in \mathbb{R}\}$$

of constant functions, while the image of T is the infinite dimensional vector space

$$\mathrm{im}(T) = \{f \in V \,:\, f(0) = 0\}$$

of smooth functions that vanish at $x = 0$.

Proof. First we'll verify that $T : V \to V$ is linear. Let $f, g \in V$. Then for all $x \in (-1, 1)$, we have

$$\begin{aligned} T(f+g)(x) &= x\,(f+g)'(x) = x\,(f' + g')(x) \\ &= x\, f'(x) + x\, g'(x) \\ &= T(f)(x) + T(g)(x) \end{aligned}$$

and thus $T(f + g) = T(f) + T(g)$ as functions. Now let $f \in V$ and $c \in c \in \mathbb{R}$. Then for all $x \in (-1, 1)$, we have

$$\begin{aligned} T(cf)(x) &= x\,(cf)'(x) = x\, cf'(x) = c\,(x\, f'(x)) \\ &= (c\, T(f))(x) \end{aligned}$$

and thus $T(cf) = c\, T(f)$. We've just shown that $T : V \to V$ is a linear transformation.

Now for the kernel. First note that any constant function is in the kernel of T. Now we'll show that the kernel only consists of constant functions. Let $f \in V$ and suppose $f \in \ker(T)$. Then

$$x\, f'(x) = 0,\ \forall x \in (-1, 1).$$

For $x \neq 0$ we can divide by x, and thus

$$f'(x) = 0,\ \forall x \in (-1, 1),\ x \neq 0.$$

but at $x = 0$, a priori $f'(0)$ could be **anything**! But in fact, since f is infinitely differentiable (at $x = 0$), f' is then differentiable, hence continuous (at $x = 0$). Thus for any sequence (x_n) such that $x_n \in (-1, 1)$, $x_n \neq 0$, and $x_n \to 0$, the sequence $(f'(x_n))$ converges

$f'(x_n) \to f'(0)$, since f' is continuous. But $f'(x_n) = 0$ for all n, so the zero sequence tends to $f'(0)$, i.e. $f'(0) = 0$. Thus we can finally say what we always wanted to say,

$$f'(x) = 0, \ \forall x \in (-1, 1),$$

and hence by calculus, f is a constant function on the interval $(-1, 1)$.

Now for the image. First notice that for any $f \in V$,

$$T(f)(0) = 0 \cdot f(0) = 0,$$

and thus $\mathrm{im}(T) \subset \{f \in V \ : \ f(0) = 0\}$. Now we need to prove that every $f \in V$ such that $f(0) = 0$ is equal to $T(g)$ for some $g \in V$, i.e. $f(x) = x\, g'(x)$. So let $f \in V$ and assume that $f(0) = 0$. The idea is to take an antiderivative of "$\frac{1}{x} f(x)$". We first need to know that this makes sense, and then that it exists in V. Define a function $F : (-1, 1) \to \mathbb{R}$ by

$$F(x) = \begin{cases} \frac{1}{x} f(x) & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

First we'll claim that F is continuous. It's certainly continuous (the product of two continuous functions) for $x \neq 0$, the problem is at $x = 0$. But now note that

$$\lim_{x \to 0} F(x) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = F(0),$$

by the definition of the derivative of f at $x = 0$. Thus F is continuous at $x = 0$. Also note that $x\, F(x) = f(x)$ for all $x \in (-1, 1)$, including at $x = 0$. Finally, define $g : (-1, 1) \to \mathbb{R}$ to be an antiderivative of F,

$$g(x) = \int_{t=0}^{x} F(t)\, dt.$$

Now we need to show that $g \in V$, i.e. that g is smooth. Once we've shown that, then we can verify that in fact $T(g) = f$. And we'll have shown that $\mathrm{im}(T) = \{f \in V \ : \ f(0) = 0\}$.

Now about showing g is smooth. Since by the fundamental theorem of calculus, $Dg = F$, we are reduced to showing that F is smooth. It's certainly smooth (a product of smooth functions) at all $x \neq 0$, so the problem is at $x = 0$. This is a bit tricky. A nice way is to use the Taylor expansion of f around $x = 0$, noting that since $f(0) = 0$, there is no constant term, and you can "divide by x". Staring at the Taylor expansion, you'll notice that there's a nice formula,

$$F^{(n)}(0) = \frac{1}{n+1} f^{(n+1)}(0), \quad \text{for all } n \geq 0.$$

for the higher derivatives of F at $x = 0$. Another way, without using a Taylor expansion, is to actually compute, via the limit definition, the higher derivatives of F at $x = 0$, and to prove (perhaps with the help of l'Hôpital's rule and induction) that they exist and match the above formula. For example, to compute the first derivative,

$$\begin{aligned} F'(0) &= \lim_{x \to 0} \frac{\frac{1}{x} f(x) - f'(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x\, f'(x) - f(x)}{x^2} \\ &= \lim_{x \to 0} \frac{x\, f''(x) + f'(x) - f'(x)}{2x} = \frac{1}{2} f''(x) \end{aligned}$$

using l'Hôpital's rule twice. Thus we find that the first derivative $F' : (-1, 1) \to \mathbb{R}$ exists and is defined by

$$F'(x) = \begin{cases} \frac{x\,f'(x) - f(x)}{x^2} & x \neq 0 \\ \frac{1}{2} f''(0) & x = 0 \end{cases}.$$

Now we can calculate $F''(0)$ in the same way. Setting this up for a proof by induction should be doable, and perhaps fun!

Finally, note that $\operatorname{im}(T) = \{f \in V \ : \ f(0) = 0\}$ contains all polynomials x^n for $n \geq 1$, since they all vanish at $x = 0$. These certainly span an infinite dimensional subspace of $\operatorname{im}(T)$, which must thus be infinite dimensional itself. Alternatively, we could appeal to exercise 2.4 #30. $\square$

As you might have noticed, there are quite a lot of interesting aspects to this problem that unfortunately **nobody** in the class touched on.

2.4 #30 For the next proposition, I'll provide two proofs. One will be a quick proof by contradiction, and the second will be an alternate proof of the contrapositive statement. More people employed a variant of the first proof.

Proposition 2. *Let $T : V \to W$ be a linear transformation. Then if V is infinite dimensional then* $\ker(T)$ *or* $\operatorname{im}(T)$ *is infinite dimensional (or both).*

Proof (by contradiction). Let V be an infinite dimensional vector space. To get a contradiction, assume that both $\ker(T)$ and $\operatorname{im}(T)$ are finite dimensional, say of respective dimensions k and r. Since $\ker(T) \subset V$ is a finite dimensional subspace, let $e_1, \ldots, e_k \in V$ be a basis for $\ker(T)$. Choose an integer $n > r$ and, since V is assumed infinite dimensional, we can keep choosing vectors $e_{k+1}, \ldots, e_{k+n} \in V$ such that $e_1, \ldots, e_{k+n}$ are linearly independent. Consider the n vectors $T(e_{k+1}), \ldots, T(e_{k+n}) \in \operatorname{im}(T)$. Since $n > r$ and the dimension of $\operatorname{im}(T)$ is assumed to be r, these must be linearly dependent, i.e. there exist scalars $c_{k+1}, \ldots, c_{k+n} \in \mathbb{R}$ not all zero such that

$$0 = c_{k+1}T(e_{k+1}) + \cdots + c_{k+n}T(e_{k+n}) = T(c_{k+1}e_{k+1} + \cdots + c_{k+n}e_{k+n})$$

by the linearity of T. Thus we see that the vector $c_{k+1}e_{k+1} + \cdots + c_{k+n}e_{k+n}$ is in the kernel of T, and thus has a unique representation $v = c_1e_1 + \cdots + c_ke_k$ with respect the the chosen basis of $\ker(T)$. Thus we have the equation,

$$c_1e_1 + \cdots + c_ke_k = c_{k+1}e_{k+1} + \cdots + c_{k+n}e_{k+n}$$

and hence we've found a linear dependence between $e_1, \ldots, e_{k+n}$, which is a contradiction to the fact that we could choose them linearly independent. So we're done. $\square$

Proof (of contrapositive). The contrapositive of the proposition is:

> Let $T : V \to W$ be a linear transformation. Then if $\ker(T)$ and $\operatorname{im}(T)$ are finite dimensional, then so is V.

So assume that $\ker(T)$ and $\operatorname{im}(T)$ are both finite dimensional, say of respective dimensions k and r. Choose a basis $e_1, \ldots, e_k \in V$ of $\ker(T)$. Choose a basis $f_1, \ldots, f_r \in \operatorname{im}(T)$ of $\operatorname{im}(T)$, and choose a elements $e_{k+1}, \ldots, e_{k+r} \in V$ such that

$$T(e_{k+i}) = f_i, \quad \text{for all } i = 1, \ldots, r.$$

We can do this by the definition of $\operatorname{im}(T)$. First we claim that $e_1, \ldots, e_{k+r}$ are linearly independent. To that end, suppose that for some scalars $c_1, \ldots, c_{k+r} \in \mathbb{R}$ we have,

$$c_1 e_1 + \cdots + c_{k+r} e_{k+r} = 0.$$

We want to show that they must all be zero. By the linearity of T, we see that

$$\begin{aligned} 0 &= T(c_1 e_1 + \cdots + c_{k+r} e_{k+r}) \\ &= c_1 T(e_1) + \cdots + c_{k+r} T(e_{k+r}) \\ &= c_{k+1} f_1 + \cdots + c_{k+r} f_r \end{aligned}$$

since $T(e_i) = 0$ for $i = 1, \ldots, k$. Thus $c_{k+1} = \cdots = c_{k+r} = 0$ since $f_1, \ldots, f_r \in \operatorname{im}(T)$ are linearly independent. But now,

$$0 = c_1 e_1 + \cdots + c_{k+r} e_{k+r} = c_1 e_1 + \cdots + c_k e_k,$$

and thus $c_1 = \cdots = c_k = 0$ as well, since $e_1, \ldots, e_k \in V$ are linearly independent. Thus $c_1 = \cdots = c_{k+r} = 0$. Thus we're proved that $e_1, \ldots, e_{k+r} \in V$ are linearly independent.

Now we want to prove that this forms a basis, i.e. that $e_1, \ldots, e_{k+r}$ is a maximal linearly independent set. To that end, choose any $v \in V$. Then $T(v) \in \operatorname{im}(T)$, so we can uniquely express

$$T(v) = c_{k+1} f_1 + \cdots + c_{k+r} f_r = c_{k+1} T(e_{k+1}) + \cdots + c_{k+r} T(e_{k+r}),$$

in terms of our chosen basis of $\operatorname{im}(T)$. Now note that

$$0 = T(v) - c_{k+1} T(e_{k+1}) + \cdots + c_{k+r} T(e_{k+r}) = T(v - c_{k+1} e_{k+1} + \cdots + c_{k+r} e_{k+r})$$

and so the element $v - c_{k+1} e_{k+1} + \cdots + c_{k+r} e_{k+r}$ is in the kernel of T, so can be uniquely expressed as

$$v - c_{k+1} e_{k+1} + \cdots + c_{k+r} e_{k+r} = c_1 e_1 + \cdots + c_k e_k.$$

Finally we see that $v \in V$ is dependent on $e_1, \ldots, e_{k+r}$. Thus we've found a maximal linearly independent set, i.e. a basis, so that V is finite dimensional, with dimension $k + r$, just as it should be. Incidentally, we've given an alternate proof of the rank-nullity theorem. $\square$

Notice that the two proofs differ in very specific ways, though have much in common. In general, it is more "tasteful" to prove things by proving their contrapositive, though it is often "easier" to prove things by contradiction. So my rule of thumb is that if I can get by with the former, I'll avoid the later.