Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2015

Final Exam Review Sheet

Directions: The final exam will take place in ML 211 (Mason Laboratory, 9 Hillhouse Avenue) on Friday, December 18th from 2:00–5:30 pm. You will have 3 hours to complete the exam and 30 minutes to review your answers. No electronic devices will be allowed. No notes will be allowed. On all problems, you will need to write your thoughts/proofs in a coherent way to get full credit.

Topics covered and practice problems:

- Basic set theory and functions (injections, surjections, bijections). Definition of a group. Modular arithmetic. DF 0.1 Exercises 4–6; DF 0.2 Exercises 7, 8, 11; DF 0.3 Exercises 4, 5, 7, 9; DF 1.1 Exercises 1, 2, 5–9, 12–14, 21, 28, 31, 36.

- Dihedral groups. Symmetric groups. Disjoint cycle decomposition of permutations. Matrix groups over a field. Alternating groups. Conjugacy classes in S_n. DF 1.2 Exercises 4–6, 9; DF 1.3 Exercises 9–19; DF 1.4 Exercises 7, 10; DF 3.5 Exercises 2–6, 9–12, 15–16.

- Homomorphisms and isomorphisms. Kernel. Image. DF 1.6 Exercises 3–9, 11, 15–16, 19, 21–22, 25.

- Group actions. Permutation representation. Kernel. Faithful. Transitive. Orbit. Stabilizer. Left multiplication action. Conjugation action. Conjugacy classes. Class equation. p-groups have nontrivial center. Groups of order p^2 are abelian. DF 1.7 Exercises 1–3, 5–6, 8–13, 20, 21, 23; DF 4.1 Exercises 1–6; DF 4.2 Exercises 1–6, 10, 13; DF 4.3 Exercises 2–3, 6–12, 25, 28–29.

- Subgroups. Centralizers. Normalizers. Cyclic subgroups. Generators. Lattice of subgroups. DF 2.1 Exercises 1–5, 14; DF 2.2 Exercises 1–2, 7, 10–11; DF 2.3 Exercises 1–5, 9, 11–13, 15; DF 2.4 Exercises 6–9, 18; DF Exercises 4, 6–10;

- Quotient groups. Cosets. Isomorphism theorems. Composition series. Simple groups. A_5 is simple. Solvable groups. DF 3.1 Exercises 6–13, 31, 33–34; DF 3.2 Exercises 4, 8, 13–16, 22–23; DF 3.3 Exercises 1, 3, 8; DF 3.4 Exercises 1–2, 5, 7.

- Automorphism groups. Inner automorphisms. DF 4.4 Exercises 1, 3, 5, 12–14, 15, 16.

- Sylow p-subgroups. Sylow's Theorem. Number of Sylow p-subgroups. Applications to groups of small order: pq, pqr, $2^k \cdot 3$, $4 \cdot 3^k$, 56. DF 4.5 Exercises 5–9, 13–30, 39–40, 54–55.

- Direct and semidirect products. Recognition theorem for direct and semidirect products. Classification of groups of small order: pq, 12, 30, p^3. Holomorph groups. Commutator subgroup. Abelianization. Fundamental theorem of finitely generated abelian groups. Invariant factors and elementary divisors. DF 5.1 Exercises 1, 4, 9, 11, 14, 18. DF Exercises 1–6, 9, 10, 16; DF 5.3 Exercise 1; DF 5.4 Exercises 1, 4, 5, 7, 10–14; DF 5.5 Exercises 1, 4–15, 17, 20, 22, 24.

- Rings. Division rings. Quaternion rings. Quadratic fields. Quadratic integer rings. Matrix rings. Polynomial rings. Group rings. Subrings. Zero divisors. Nilpotent elements. Group of units. Integral domains. Finite integral domain is a field. DF 7.1 Exercises 3–6, 8, 12, 13, 14, 17, 24, 25, 29; DF 7.2 Exercises 1–4, 5a, 6, 9–11.

- Ring homomorphisms. Ideals. Quotient rings. Isomorphism theorems for rings. Prime ideals. Maximal ideals. Rings of fractions. Chinese remainder theorem. DF 7.3 Exercises 1–14, 18–21, 23, 24–26, 28, 31–32, 34, 36–37; DF 7.4 Exercises 4–9, 11, 13–19, 22, 27; DF 7.5 Exercises 2–5; DF 7.6 Exercises 3, 4, 5ac, 7.

- Euclidean domains. Principal ideal domains. Unique factorization domains. Prime and irreducible elements. $\mathbb{Z}$ and $F[x]$ are Euclidean. Euclidean $\Rightarrow$ PID. PID $\Rightarrow$ UFD. Examples showing that each of these implications cannot be reversed (e.g., quadratic integer rings, $\mathbb{Z}[x]$, $F[x, y]$, etc.). R UFD $\Leftrightarrow$ $R[x]$ UFD. $F[x_1, \ldots, x_n]$ UFD. DF 8.1 Exercises 1–7, 10, 11; DF 8.2 Exercises 1, 3, 5, 8; DF 8.3 Exercises 1, 2, 5, 6, 8; DF 9.1 Exercises 1, 2, 4–7, 9, 13, 14; DF 9.2 Exercises 1–10; DF 9.3 Exercises 1, 3, 4.

- Modules. Submodules. Module homomorphisms. Quotient modules. Isomorphism theorems. F-modules are F-vector spaces. $\mathbb{Z}$-modules are abelian groups. $F[x]$-modules are F-vector spaces with a choice of linear operator. F-algebras. Generators. Direct sums. Free modules. Cyclic modules. DF 10.1 Exercises 4–8, 17–21; DF 10.2 Exercises 1, 3–12; DF 10.3 Exercises 2–4, 6.

- Modules over a PID. Linear independence. Rank. Torsion submodule. Invariant factors. Elementary divisors. DF 12.1 Exercises 1, 3–7, 13.

Practice exam questions:

1. There will be some True/False questions covering a range of topics.

2. There will be some problems of the form: Given a subset of a group, determine whether it is a subgroup; given a subset of a ring, determine whether it is subring or ideal; and/or given a subset of a module, determine whether it is a submodule.

3. Show that every finite group is isomorphic to a subgroup of A_n for some n.

4. Let p, q, r be prime numbers (not necessarily distinct). Depending on the values of p, q, r, determine the number of abelian groups of order $(pqr)^2$.

5. Classify all groups of order 245 up to isomorphism.

6. Show that if G is a group of order $7!$ with a normal subgroup H isomorphic to S_6, then $G \cong Z_7 \times S_6$. Conclude that S_7 does not have a normal subgroup isomorphic to S_6.

7. Construct fields of order 8, 27, and 125. What are the possible (multiplicative) orders of nonzero elements in those fields?

8. Consider the subset R of $M_2(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Prove that R is a subring of $M_2(\mathbb{R})$. Prove that R is a commutative ring with 1, but is not an integral domain. Find all idempotents in R (an idempotent is an element x such that $x^2 = x$). Find all nilpotent elements in R. Define $\varphi : R \to \mathbb{R}$ by

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto a - b.$$

Show that φ is a ring homomorphism. Determine the kernel of φ as well as $R/\ker(\varphi)$. Is $\ker(\varphi)$ a prime ideal or a maximal ideal?

9. Let R be the ring of $\mathbb{Z}$-quaternions. Compute the group $R^\times$.

10. Find a unit of every degree in $\mathbb{Z}/4\mathbb{Z}[x]$.

11. Compute the structure (invariant factors or elementary divisors) of the group of $\mathbb{Z}$-module homomorphisms $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/36\mathbb{Z}, \mathbb{Z}/48\mathbb{Z})$. Identify, as best you can, the rings $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/36\mathbb{Z}, \mathbb{Z}/36\mathbb{Z})$ and $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}/48\mathbb{Z}, \mathbb{Z}/48\mathbb{Z})$.

12. Give an example of the following or prove that none exists:

- A Euclidean domain other than $\mathbb{Z}$ and $F[x]$.
- A PID other than $\mathbb{Z}$ and $F[x]$.
- A quotient of $\mathbb{Z}$ or $F[x]$ that is not Euclidean.
- A quotient of $\mathbb{Z}$ or $F[x]$ that is not a PID.
- A PID that is not Euclidean.
- A UFD that is not a PID.
- A UFD that is a Euclidean domain but is not a PID.

13. Find all integers with a remainder of 1 when divided by 2, 3, and 5. Find all integers with a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, and a remainder of 3 when divided by 5.

14. Let the symmetric group S_3 act on $\mathbb{R}^3$ by permuting the standard basis. There is an induced $\mathbb{R}[S_3]$-module structure on $\mathbb{R}^3$. Find a nontrivial $\mathbb{R}[S_3]$-submodule $N \subset \mathbb{R}^3$.

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