Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2015

Problem Set # 6 (due at the beginning of class on Friday 6 November)

Notation: If G is a group and $G' = [G,G]$ is the commutator subgroup, then $G^{\text{ab}} = G/[G,G]$ is the **abelianization** of G. Then G^{ab} is abelian and there is a canonical surjective homomorphism $G \to G^{\text{ab}}$, which is often also called the abelianization.

Let $H \trianglelefteqslant G$ and $\pi : G \to G/H$ be the natural projection. The following is known as the **universal property of the quotient**: if $\phi : G \to K$ is a group homomorphism such that $H \subset \ker(\phi)$ then there exists a unique homomorphism $F : G/H \to K$ such that $F \circ \pi = \phi$.

Reading: DF 4.5, 5.1–5.4.

Problems: (Starred* problems are strongly recommended!)

1. DF 5.1 Exercises 4, 8, 17* (see 15 for notation).

2. DF 5.2 Exercises 2, 3, 6, 7*, 8, 9, 11, 14*.

3. DF 5.4 Exercises 4, 5*, 7, 11, 19*.

4. *Unfinished business.** The goal of this problem is to complete the proof that 60 is the least order of a finite simple nonabelian group. From class, we had eliminated most of the orders up to 60 using the following facts:

- Every p-group has nontrivial center. (We proved this using the class equation.)
- Every group of order $n = p^k m$ with $p \nmid m$ and $m < p$, has a normal Sylow p-subgroup. (From the Sylow theorems.)

This left us with orders 12, 24, 30, 36, 40, 45, 48, and 56. We then eliminated case by case:

- We eliminated 40 and 45 using the congruence and divisibility conditions on the number of Sylow subgroups, finding normal Sylow subgroups.
- We eliminated 30 using the fact that every group whose order is the product of three distinct primes has a normal Sylow subgroup. (This is DF §4.3 Problem 16.)
- We eliminated 12 by appealing to a result proved in the book: every group of order 12 either has a normal Sylow 3-subgroup or is isomorphic to A_4. (See DF §4.5, p. 144.)

This left us with orders 24, 36, 48, and 56. Prove the following to complete our goal:

(a) If G is a finite group of order $2^k \cdot 3$, with $k \geq 2$, then G is not simple.
(b) If G is a finite group of order $2^2 \cdot 3^k$, with $k \geq 2$, then G is not simple.
(c) If G is a finite group of order 56 then G is not simple.

Hint: For the first two, consider the conjugation action on the Sylow 2- or 3-subgroups, respectively. For the third, if neither the Sylow 2- nor 7- subgroups are normal, start counting elements in these subgroups to reach a contradiction (note that any two Sylow 7-subgroups only intersect at the identity, how could Sylow 2-subgroups intersect?).

5. *Abelianizing.** Prove the following:

(a) Let $\phi : G \to G^{\text{ab}}$ be the canonical surjection. For any abelian group H and any homomorphism $f : G \to H$ there exists a unique homomorphism $F : G^{\text{ab}} \to H$ such that $f = F \circ \phi$. This is called the **universal property of abelianization**. Hint: Use another universal property.
(b) For any groups H and K, there is an isomorphism $(H \times K)^{\text{ab}} \cong H^{\text{ab}} \times K^{\text{ab}}$.
(c) Prove that S_4 is not isomorphic to the direct product $H \times K$ of nontrivial groups. Hint: Abelianize both sides.