Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2017

Problem Set # 2 (due at the beginning of class on Friday 22 September)

Notation: Z_n is an abstract cyclic group written multiplicatively.

Reading: DF 1.4, 1.6, 2.1, 2.3.

Problems:

1. DF 1.6 Exercises 2*, 3, 4*, 6*, 7, 9* (here D_{24} is the dihedral group with 24 elements), 14*, 16, 17* (prove that it's always a bijection), 18, 24*, 25.

2. DF 2.1 Exercises 2, 6*, 7, 8, 9*, 10, 12, 14.

3. DF 2.3 Exercises 2, 5, 8*, 10, 11, 12*, 13, 20, 21*, 22*, 23* (Hint: What does 22 tell you about the order of 5 in $(\mathbb{Z}/2^n\mathbb{Z})^\times$?), 25, 26*.

4. *Fields of order 4.*

(a) Let $F = \{0, 1, x, y\}$. Prove that there are operations $+$ and $\cdot$ on F, such that $1 + x = y$ and $x^2 = y$, making F into a field. (Note that the four elements of F are distinct!) Essentially the problem is to fill out the addition and multiplication tables:

$+$	0	1	x	y
0				
1				
x				
y				

$\cdot$	0	1	x	y
0				
1				
x				
y				

You already know certain rows and columns by properties of 0 and 1 in a field!

(b) Let F_1 and F_2 be fields. A map $\phi : F_1 \to F_2$ is an **isomorphism of fields** if ϕ is a bijection satisfying $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ and $\phi(1_{F_1}) = 1_{F_2}$. An isomorphism between a field and itself is called an **automorphism**. Find a non-identity automorphism of the field F of order 4 described above.

(c) Let F' be any field with 4 elements. Prove that there exists an isomorphism $\phi : F \to F'$, where F is the field described above.

This shows that there is a unique "isomorphism class" of field of order 4, which we call $\mathbb{F}_4$.

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