Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2017

Problem Set # 6 (due at the beginning of class on October 27, have a restful Fall break!)

Reminder. On a previous problem set, you proved that the map $(\mathbb{Z}/n\mathbb{Z})^\times \to \mathrm{Aut}(Z_n)$, defined by $a \mapsto \phi_a$ where $\phi_a(x) = x^a$ for $x \in Z_n$, is an isomorphism.

Reading: DF 3.5, 4.3–4.5.

Problems:

1. DF 3.5 Exercises 3, 4*, 5, 6, 14, 15, 17*.

2. DF 4.3 Exercises 3, 5*, 6*, 8, 13, 19* (you do not need to draw the lattice), 21, 22, 28* (if $P \leqslant G$ is any subgroup, use the left multiplication action of G on the set of cosets G/P, see pp. 118–119), 29, 31, 32, 34*.

3. DF 4.4 Exercises 1, 3*, 5*, 10, 11*, 18*, 19.

4. DF 4.5 Exercises 3 (only use the existence of Sylow p-subgroups), 4–7, 8.

5. Let V is a vector space over a field, and recall that from the previous problem set we have an injective homomorphism $\mathrm{GL}(V) \to \mathrm{Aut}(V, +)$ that is an isomorphism when the field is $\mathbb{F}_p$. Now let V be a 2-dimensional vector space over the field $\mathbb{F}_4$, and prove that $\mathrm{GL}(V) \to \mathrm{Aut}(V, +)$ is not surjective. Hint: Calculate the orders of $\mathrm{GL}(V)$ and $\mathrm{Aut}(V, +)$.

6. Let $g_1, \ldots, g_r$ be a set of representatives of *all* conjugacy classes (including central ones) of a finite group G. Then the equation $|G| = |\mathcal{K}_{g_1}| + \cdots + |\mathcal{K}_{g_r}|$ transforms, after dividing both sides by $|G|$, into an *Egyptian fraction*

$$1 = \frac{1}{|C_G(g_1)|} + \cdots + \frac{1}{|C_G(g_r)|}.$$

For example, the group S_3 gives $1 = \frac{1}{6} + \frac{1}{3} + \frac{1}{2}$. For each of the following Egyptian fractions, determine if it arises from a finite group G. If so, write G; if not, explain why not.

(a) $1 = \dfrac{1}{3} + \dfrac{1}{3} + \dfrac{1}{3}$

(b) $1 = \dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{2}$

(c) $1 = \dfrac{1}{6} + \dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{3}$

(d) $1 = \dfrac{1}{8} + \dfrac{1}{8} + \dfrac{1}{4} + \dfrac{1}{2}$

(e) $1 = \dfrac{1}{8} + \dfrac{1}{8} + \dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{4}$

(f) $1 = \dfrac{1}{12} + \dfrac{1}{4} + \dfrac{1}{3} + \dfrac{1}{3}$

(g) $1 = \dfrac{1}{12} + \dfrac{1}{12} + \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{4} + \dfrac{1}{4}$

(h) $1 = \dfrac{1}{24} + \dfrac{1}{8} + \dfrac{1}{4} + \dfrac{1}{4} + \dfrac{1}{3}$

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