Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2017

Problem Set # 8 (due at the beginning of class on Friday 10 November)

Notation: If G is a group then the commutator subgroup $G' = [G, G]$ is the subgroup generated by all commutators $xyx^{-1}y^{-1}$ for $x, y \in G$. Then $[G, G] \trianglelefteq G$ and we call $G^{\mathrm{ab}} = G/[G, G]$ is the **abelianization** of G. In fact, G^{ab} is abelian and the canonical projection $G \to G^{\mathrm{ab}}$ is a surjective homomorphism often also called the abelianization.

Let $K \trianglelefteq G$ and $\pi : G \to G/K$ be the natural quotient map. The following is known as the **universal property of the quotient**: if $\xi : G \to H$ is a group homomorphism such that $K \subseteq \ker(\xi)$ then there exists a unique homomorphism $F : G/K \to H$ such that $F \circ \pi = \xi$.

Reading: DF 5.2–5.5.

Problems:

1. DF 5.2 Exercises 2, 3, 5, 6, 7, 8*, 9, 11, 14*. The notions of **exponent**, **rank**, and **free rank** are defined just above the exercise section.
Hint: For 8(b), in the hinted at reduction to elementary abelian groups, use part (a) and the 4th isomorphism theorem; then recall that it is often fruitful to think of elementary abelian p-groups as finite dimensional vector spaces over $\mathbb{F}_p$, cf. Problem Set #5.

2. DF 5.4 Exercises 4, 5*, 7, 19*.

3. DF 5.5 Exercises 5, 6*, 7*, 8, 10*, 11*, 13*, 23.

4. Prove that if n is odd, then D_{4n} is isomorphic to $D_{2n} \times Z_2$. Prove that D_{8n} is not isomorphic to $D_{4n} \times Z_2$.

5. *Abelianizing.* Prove the following:

(a) The **universal property of abelianization**: Let $\phi : G \to G^{\mathrm{ab}}$ be the abelianization. For any abelian group H and any homomorphism $f : G \to H$ there exists a unique homomorphism $F : G^{\mathrm{ab}} \to H$ such that $f = F \circ \phi$. Note. This is proved in DF 5.4, but the proof there is a bit long-winded. Use the universal property of the quotient instead.

(b) For any groups H and K, there is an isomorphism $(H \times K)^{\mathrm{ab}} \cong H^{\mathrm{ab}} \times K^{\mathrm{ab}}$. Hint. Use the universal property.

(c) Prove that S_4 is not isomorphic to the direct product $H \times K$ of nontrivial groups. Hint: Abelianize both sides.