Yale University Department of Mathematics
Math 370 Fields and Galois Theory
Spring 2018

Problem Set # 10 (due in class on Thursday April 26)

Notation: A field is algebraically closed if every nonconstant polynomial has a root.

Reading: GT 15.

Problems:

1. GT Exercise 15.6. This shows the subtletly inherent in our notion of "radical" extension. It is quite subtle to come up with an example of a radical extension with a subextension that is not radical. **Miki's hints.** Use $x^7 - 1$. Do this very carefully. A simple extension $F(\alpha)$ can be radical even if α is not an nth root of anything in F (e.g., $\mathbb{Q}(\omega)$). It would be very hard to prove that *no generator* is an nth root, so you need to find a different way to prove that a given extension is not radical. Finally, do this very very carefully.

2. GT Exercise 15.7. This is false if the polynomial is not irreducible; why?

3. Let p be prime and $q = p^n$. Prove that $x^q - x \in \mathbb{F}_p[x]$ factors as the product of all distinct monic irreducible polynomials of degree dividing n over $\mathbb{F}_p$.

4. *Finite subgroups of fields.* Let F be a field. Understand at least three proofs, and then provide your favorite one, of the fact that every finite subgroup of the multiplicative group $F^\times$ is cyclic. For inspiration, see this MathOverflow post.

5. *Fundamental Theorem of Algebra.* An **ordered field** is a field F together with a subset F^+ of **positive elements** satisfying: $a, b \in F^+ \Rightarrow a + b \in F^+$ and $ab \in F^+$ and for each $a \in F$ exactly one of $a \in F^+$, $a = 0$, or $-a \in F^+$ is true.

(a) Prove that if F is an ordered field then any nonzero square is positive, that -1 is not positive, and that F has characteristic zero. Also, prove that $F(i) = F[x]/(x^2 + 1)$ is not an ordered field. **Challenge.** Prove that a field F can be ordered if and only if -1 is not a sum of squares.

(b) An ordered field F is called **real closed** if every positive element has a square root and every polynomial of odd degree over F has a root. Prove that $\mathbb{R}$ and $\mathbb{R} \cap \overline{\mathbb{Q}}$ are real closed. **Hint.** You may need a tiny bit of analysis, but try to keep it to a minimum.

(c) Prove that a real closed field does not have any nontrivial finite extensions of odd degree.

(d) Prove that if F is real closed then the only quadratic extension of F is $F(i)$, and every element of $F(i)$ has a square root.

(e) Prove that a field K is algebraically closed if and only if it does not admit any nontrivial algebraic extensions if and only if it does not admit any nontrivial finite extension.

(f) Prove that if F is a real closed field then $F(i)$ is algebraically closed. **Hint.** First, let $L'/F(i)$ be a finite extension and L/F the normal closure of L'/F. Then why is L/F a Galois extension whose group G has even order? Let $H \subset G$ be a Sylow 2-subgroup. Use the Galois correspondence with $H \subset G$ to prove that G is actually a 2-group. Remember the result from abstract algebra that every finite p-group has a subgroup of index p, and use this, with the Galois correspondence, to prove that actually G must be trivial.

(g) Deduce that $\mathbb{C}$ and $\overline{\mathbb{Q}}$ are algebraically closed.

6. Let $\alpha \in \mathbb{C}$ be algebraic of degree 4 over $\mathbb{Q}$. Prove that α is constructible if and only if the normal closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$ has Galois group C_4, V_4 (Klein four), or D_8. Soon we'll see how to write down an explicit example that is not constructible.