Yale University Department of Mathematics
Math 370 Fields and Galois Theory
Spring 2018

Problem Set # 9 (due in class on Thursday April 19)

Notation: You can use the Primitive Element Theorem, which we will prove next week, which states that every finite separable extension is simple.

Reading: GT 12, 13.

Problems:

1. Let K/F be a Galois extension with group S_3.

(a) Prove that there exists an irreducible polynomial $f(x) \in F[x]$ of degree 6 whose splitting field is K.

(b) Assume that the characteristic of F is not 2. Prove that there exists an irreducible polynomial $f(x) \in F[x]$ of degree 3 whose splitting field is K.
Remark. The cubic is more intuitive than the sextic, though slightly harder to prove.

2. Give an example of a finite extension of fields that is not simple. (It had to come eventually.)

3. More about prime cyclotomic extensions.

(a) Let p be a prime number. What is the Galois group of the splitting field $K/\mathbb{Q}$ of the polynomial $x^p - 1 \in \mathbb{Q}[x]$?

(b) In the cases $p = 5$ and $p = 7$, compute simple generators for each subfield, prove that each is normal over $\mathbb{Q}$, express each as the splitting field of an irreducible polynomial over $\mathbb{Q}$, and draw the lattices of subfields and subgroups of the Galois group.

(c) Find a Galois extension $L/\mathbb{Q}$ whose Galois group is cyclic of order 5 and an irreducible polynomial of degree 5 over $\mathbb{Q}$ whose splitting field is L. **Hint.** You can use cyclotomics.

4. Let $L/\mathbb{Q}$ be a Galois extension whose Galois group is cyclic of order 4. Prove that its unique quadratic subextension $K/\mathbb{Q}$ is real (i.e., $K = \mathbb{Q}(\sqrt{d})$ with $d > 0$). **Hint.** Complex conjugation.

5. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible quartic polynomial and $K = \mathbb{Q}(\alpha)$, where α is a root of $f(x)$. Let $G \subset S_4$ be the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that $K/\mathbb{Q}$ has no nontrivial intermediate subfields if and only if $G = A_4$ or $G = S_4$.