Yale University Department of Mathematics
**Math 370 Fields and Galois Theory**
Spring 2019

Problem Set # 3 (due in class on Thursday 7 February)

**Notation:** Let $F$ be a field. A field extension $K/F$ is **simple** if $K = F(\alpha)$ for some $\alpha \in K$ and is **finitely generated** if $K = F(\alpha_1, \dots, \alpha_n)$ for finitely many elements $\alpha_1, \dots, \alpha_n \in K$.

**Reading:** FT pp. 14–19.

**Problems:**

**1.** Let $F$ be a field of characteristic $\neq 2$ (i.e., $2 \neq 0$ in $F$) and let $K/F$ be a field extension of degree 2. Prove that there exists $\alpha \in K$ with $\alpha^2 \in F$ such that $K = F(\alpha)$.

**2.** For each extension $K/F$ and each element $\alpha \in K$, find the minimal polynomial of $\alpha$ over $K$ (and prove that it is the minimal polynomial).

(a) $i$ in $\mathbb{C}/\mathbb{R}$ (b) $i$ in $\mathbb{C}/\mathbb{Q}$ (c) $(1+\sqrt{5})/2$ in $\mathbb{R}/\mathbb{Q}$ (d) $\sqrt{2+\sqrt{2}}$ in $\mathbb{R}/\mathbb{Q}$

**3.** Let $\pi \in \mathbb{R}$ be the area of a unit circle and let $\alpha = \sqrt{\pi^2 + 2}$. Consider the field $K = \mathbb{Q}(\pi, \alpha)$. For the following field extensions, determine whether they are transcendental and/or algebraic and/or finite and/or simple, and if you determine the extension is simple and algebraic, find a simple generator and determine its minimal polynomial.

(a) $K/\mathbb{Q}$ (b) $K/\mathbb{Q}(\pi)$ (c) $K/\mathbb{Q}(\alpha)$ (d) $K/\mathbb{Q}(\pi + \alpha)$

**4.** Factor $x^3 + x + 1$ in $\mathbb{F}_p[x]$ for all primes $p \leq 11$.

**5.** Prove that any finitely generated (not necessarily finite) extension of a countable field is countable. Use this to deduce that $\mathbb{R}$ is not a finitely generated extension of $\mathbb{Q}$.

**6.** Let $p$ and $q$ be distinct prime numbers. Prove that $\mathbb{Q}(\sqrt{p} + \sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ and find the minimal polynomial of $\sqrt{p} + \sqrt{q}$ over $\mathbb{Q}$.

**7.** Let $a \in \mathbb{Q}$ be positive and not a square. Prove that $\mathbb{Q}(\sqrt[4]{a})$ has degree 4 over $\mathbb{Q}$.

**8.** Let $F$ be a field. We know that if $K = F(\alpha)$ and $\alpha$ has minimal polynomial $m(x)$, then $m(x)$ has a root over $K$. So over $K$, we can factor $m(x) = (x - \alpha)n(x)$. What more can we say in general about the factorization of $m(x)$ over $K$, i.e., how does $n(x)$ factor further?

(a) Let $\alpha = \cos(2\pi/9) \in \mathbb{R}$. Prove that $\alpha$ is algebraic over $\mathbb{Q}$ and find its minimal polynomial $m(x)$. (**Hint.** Try taking $e^{2\pi i/9}$ to various small powers.) Prove that $m(x)$ factors into linear factors over $\mathbb{Q}(\alpha)$. (**Hint.** One way is to divide $m(x)$ by $x - \alpha$ over the field $\mathbb{Q}(\alpha)$. The quotient will have degree 2, then use the quadratic formula.)

(b) Let $\alpha = \sqrt[3]{2} \in \mathbb{R}$. Prove that $\alpha$ is algebraic over $\mathbb{Q}$ and find its minimal polynomial $m(x)$. Prove that $m(x)$ does not factor into linear factors over $\mathbb{Q}(\alpha)$.

**9.** Prove that if $K/F$ is algebraic and $L/K$ is algebraic, then $L/F$ is algebraic. (You cannot assume that these extensions are finite.)

**10.** Let $F$ be a field of characteristic $\neq 2$.

(a) Let $a_1, \dots, a_n \in F$ be distinct elements such that no product $a_{i_1} \cdots a_{i_r}$, with distinct indices $i_j$, is a square in $F$. Prove that $K = F(\sqrt{a_1}, \dots, \sqrt{a_n})$ has degree $2^n$ over $F$.

(b) Prove that the field $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \dots)$ gotten by adjoining the square roots of all prime numbers to $\mathbb{Q}$, is an infinite degree algebraic extension of $\mathbb{Q}$.