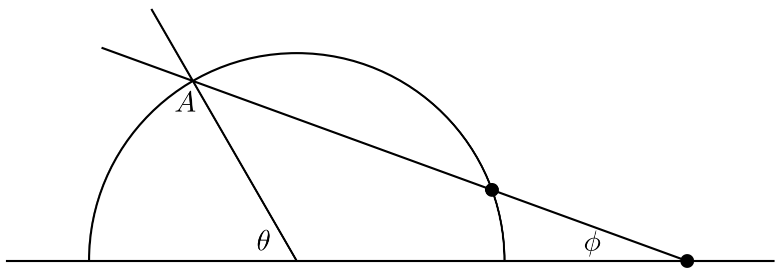
Yale University Department of Mathematics
Math 370 Fields and Galois Theory
Spring 2019

Problem Set # 4 (due in class on Thursday 14 February ♡)

Notation: For a positive integer n, write $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$.

Problems:

1. Archimedes discovered a construction that trisects any given angle using a compass and "marked" straightedge. This straightedge has two special markings a distance 1 apart. The straightedge can be placed so that it passes through any already drawn point and such that both markings intersect other already drawn lines or circles.



In the picture, start with a horizontal line and another line meeting it with angle θ. Draw a circle of radius 1 around the intersection point of these two lines. The other line meets the circle at point A. Then place the marked straightedge so that it passes through the point A and such that the markings (the dots in the picture) intersect the circle and the horizontal line. Prove that the angle ϕ that the straightedge makes with the horizontal line is equal to $\theta/3$.

2. Let p be an odd prime. Prove that both $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_{2p})$ have degree $p-1$ over $\mathbb{Q}$. Next, prove that both $\mathbb{Q}(\cos(2\pi/p))$ and $\mathbb{Q}(\cos(\pi/p))$ have degree $(p-1)/2$ over $\mathbb{Q}$.

3. Prove that any angle θ, such that $\tan(\theta)$ is rational, can be constructed with compass and straightedge.

4. Prove that an angle θ can can be trisected using compass and straightedge if and only if the polynomial $4x^3 - 3x - \cos(\theta)$ is reducible over $\mathbb{Q}(\cos(\theta))$.

5. To 5-sect an angle means to divide it by 5.

(a) Prove that the angles 2π, π, $2\pi/3$, and $\pi/2$ can be 5-sected by compass and straightedge.

(b) Prove that a general angle cannot be 5-sected by compass and straightedge.

6. About the constructibility of regular n-gons.

(a) Let p be a prime number. Prove that if a regular p-gon can be constructed, then p must be of the form $p = 2^n + 1$. **Hint.** Use ζ_p.

(b) For $2^n + 1$ to be prime, prove that n must be a power of 2.
Remark. The primes of the form $2^{2^n} + 1$ are called **Fermat primes**. The only Fermat primes we know of are $2^{2^0}+1 = 3, 2^{2^1}+1 = 5, 2^{2^2}+1 = 17, 2^{2^3}+1 = 257, 2^{2^4}+1 = 65537$. It is a big question whether there are any more!

(c) For each $3 \le n < 17$, determine whether a regular n-gon can be constructed.

7. Let K/F be a finite extension of fields and $f(x) \in F[x]$ an irreducible polynomial. Prove that if $\deg(f)$ does not divide $[K : F]$ then $f(x)$ has no roots in K.

8. For $1 \leq n \leq 8$ find the minimal polynomial $\Phi_n(x)$ of ζ_n over $\mathbb{Q}$. For each $1 \leq n \leq 8$ compute $\prod_{d|n} \Phi_d(x)$, where the product is taken over all divisors of n (including 1 and n).

9. Let F be a field and let $g(x) = x^2 + bx + c \in F[x]$. Let $K = F(\alpha)$, where α is a root of $g(x)$, so that $g(x) = (x - \alpha)(x - \beta)$ over K.

(a) Prove that $(\alpha - \beta)^2 = b^2 - 4c \in F$. This is called the **discriminant** $\Delta(g)$ of the monic quadratic polynomial g.
Hint. Use elementary symmetric polynomials.

(b) Prove that $\Delta(g) = 0$ if and only if g has repeated roots in K (i.e., if $\alpha = \beta$).

(c) Assume that the characteristic of F is not 2. Prove that $K = F(\sqrt{\Delta(g)})$. Deduce that $g(x)$ is irreducible over F if and only if $\Delta(g)$ is not a square in F. Also, prove that $g(x)$ is a square in $F[x]$ if and only if $\Delta(g) = 0$.
Hint. You are free to use the quadratic formula.

(d) Now let $F = \mathbb{F}_2(t)$ be the rational function field over $\mathbb{F}_2$. Let $g(x) = x^2 - t \in F[x]$. Prove that $g(x)$ is irreducible over F, though it satisfies $\Delta(g) = 0$. Show that $K \cong F(\sqrt{t}) := F[x]/(g(x))$ and find the roots of $g(x)$ over K. We don't know it yet, but K/F is called an inseparable quadratic extension.
Hint. Proving irreducibility is like proving $\sqrt{2}$ is irrational. For the second part, don't forget the first year's dream!

Weird stuff can happen with quadratic polynomials in characteristic 2!