Yale University Department of Mathematics
Math 373/573 Algebraic Number Theory
Spring 2019

Problem Set # 2 (due in class on Thursday 14 February ♡)

Problems:

1. *Trigonometry.*

(a) Prove that $2\sin(\pi/n)$ is an algebraic integer for all $n \geq 1$.

(b) Prove that $\sin(\pi/n)$ is not an algebraic integer for all $n \geq 3$.

2. *Inseparable.* Prove that if L/K is a finite inseparable field extension then $\mathrm{disc}(L/K) = 0$.

3. *Discriminants.* Let B be an integral domain and assume that B is a free $\mathbb{Z}$-module of rank n with $\mathrm{disc}(B/\mathbb{Z}) \neq 0$. Let $N \subset B$ be the $\mathbb{Z}$-submodule generated by elements $\beta_1, \dots, \beta_n \in B$.

(a) Prove that $D(\beta_1, \dots, \beta_n) \neq 0$ if and only if N has finite index (as abelian groups) in B, and in this case, prove that $D(\beta_1, \dots, \beta_n) = [B : N]^2 \, \mathrm{disc}(B/\mathbb{Z})$.
(Recall that $\mathrm{disc}(B/\mathbb{Z})$ can be considered as a well-defined element of $\mathbb{Z}$.)

(b) Prove that if $D(\beta_1, \dots, \beta_n)$ is squarefree (as an integer), then $\beta_1, \dots, \beta_b$ is a $\mathbb{Z}$-basis of B.
(Most often, this is applied when $\beta_i = \beta^i$ for some fixed $\beta \in B$ to deduce that $B = \mathbb{Z}[\beta]$.)

4. *The real root.* Let β be the unique positive real root of $x^3 - 4x - 1$ and let $K = \mathbb{Q}(\beta)$.

(a) Prove that $\mathcal{O}_K = \mathbb{Z}[\beta]$.

(b) Compute $\mathrm{Tr}_{K/\mathbb{Q}}(\beta^i)$ for $i = 0, 1, 2, 3$.

(c) Let $a_0, a_1, \dots$ be a sequence of integers satisfying the recursion

$$a_{n+3} - 4a_{n+1} - a_n = 0$$

for all $n \geq 0$. Prove that there are real numbers b, b', b'' such that

$$a_n = b\beta^n + b'(\beta')^n + b''(\beta'')^n$$

where β', β'' are the Galois conjugates of β.
(For inspiration, you might look up my Math 225 exercise about Fibonacci numbers, or just ask someone who took the class with me.)

(d) Determine if the limit

$$\lim_{n\to\infty} |a_n|^{1/n}$$

exists, and if so, determine it.

5. *No power.* Let $K = \mathbb{Q}(\sqrt{7}, \sqrt{10})$. Prove that there is no $\alpha \in \mathcal{O}_K$ such that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. Follow the strategy in ANT Problem 2-6 on p. 44.

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