Yale University Department of Mathematics
Math 373/573 Algebraic Number Theory
Spring 2019

Midterm Exam (due in class on 16 April)

Problems:

1. Let K be a number field and $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_K$.

(a) Prove that if there exists no element $\alpha \in \mathcal{O}_K$ with $|N_{K/\mathbb{Q}}(\alpha)| = N(\mathfrak{a})$, then $\mathfrak{a}$ is not principal.

(b) Prove that if $\alpha \in \mathfrak{a}$ satisfies $|N_{K/\mathbb{Q}}(\alpha)| = N(\mathfrak{a})$, then $\mathfrak{a} = \alpha\mathcal{O}_K$.

(c) Prove that if for $\alpha \in \mathcal{O}_K$ we have that $|N_{K/\mathbb{Q}}(\alpha)| = p_1 \cdots p_r$ is a product of distinct prime numbers, then $\alpha\mathcal{O}_K = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ is a product of prime ideals with each $\mathfrak{p}_i | p_i$ of inertial index 1.

2. Compute the class group of $\mathbb{Q}(\sqrt{-33})$. **Hint.** Consider the norms of $a + \sqrt{-33}$ for small values of $a \in \mathbb{Z}$ to get relations.

3. Let $K = \mathbb{Q}(\alpha)$ where α is the unique real root of $x^3 + 4x - 1$.

(a) Determine $\mathcal{O}_K$.

(b) Prove that the unit group $U_K = \mathcal{O}_K^\times$ is infinite. **Hint.** Find a unit $u \neq \pm 1$.

(c) Prove that $2\mathcal{O}_K = \mathfrak{p}_2\mathfrak{q}_2$ and $3\mathcal{O}_K = \mathfrak{p}_3\mathfrak{q}_3$ with $\mathfrak{p}_2, \mathfrak{p}_3$ having inertial degree 1 and $\mathfrak{q}_2, \mathfrak{q}_3$ having inertial degree 2.

(d) Prove that $\mathfrak{p}_2^2$ and $\mathfrak{p}_3^2$ are principal. **Hint.** Use Problem 1 with well-chosen elements of the form $a + \alpha^2$ for $a \in \mathbb{Z}$.

(e) How many of $\mathfrak{p}_2$, $\mathfrak{q}_2$, $\mathfrak{p}_3$, or $\mathfrak{q}_3$ can you prove are nonprincipal? **Hint.** This is not easy, and is no longer a required problem! We will learn how to do this in lecture shortly! The upshot is that one of them, equivalently (why?) all of them, are nonprincipal.

(f) Given part (e), compute the class group of K. **Hint.** Consider the norm of $\alpha + 1$ to get a relation.

4. Let p be an odd prime number and $K = \mathbb{Q}(\zeta_p)$.

(a) Prove that $N_{K/\mathbb{Q}}(1 - \zeta_p) = p$.

(b) Prove that the discriminant of $x^{p-1} + \cdots + x + 1$ is $\pm p^{p-2}$. **Hint.** The Vandermonde matrix times its transpose is a matrix of power sums. Prove that all prime-to-p powers sums of the primitive pth roots of unity are equal to -1.

(c) Determine which rational primes ramify in K and determine their decomposition into prime ideals. **Warning.** While it is true that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$, we have not proved it yet, and you cannot use it here, nor do you need it. **Hint.** Eventually, use part (a).

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