Yale University Department of Mathematics
Math 604 Introduction to Quadratic Forms
Fall 2016

Final Exam (due 5:30 pm December 21)

1. *Quadratic forms over finite fields.* Let q be an odd prime power.

(a) Show that $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ has order 2.

(b) Show than any element of $\mathbb{F}_q$ is a sum of two squares.

(c) Show that every binary quadratic form over $\mathbb{F}_q$ represents every nonzero element.

(d) Show that if $q \equiv 1 \pmod 4$ then $W(\mathbb{F}_q)$ is isomorphic to the ring $\mathbb{Z}/2\mathbb{Z}[\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}]$

(e) Show that if $q \equiv 3 \pmod 4$ then $W(\mathbb{F}_q)$ is isomorphic to the ring $\mathbb{Z}/4\mathbb{Z}$.

(f) Show that the isomorphic type of $GW(\mathbb{F}_q)$ as a ring does not depend on q.

2. *Characteristic 2, scary!* Let F be a field of characteristic 2 and $a, b \in F$. A quadratic form $q : V \to F$ is **nondegenerate** if its associated bilinear form $b_q : V \times V \to F$ defined by $b_q(v, w) = q(v + w) - q(v) - q(w)$ has a radical of dimension at most 1. Define the quadratic form $[a, b]$ on F^2 by $(x, y) \mapsto ax^2 + xy + by^2$. Let $\mathbb{H}$ be the hyperbolic form on F^2 defined by $(x, y) \mapsto xy$.

(a) Prove that $\langle a\rangle$ is nondegenerate for any $a \in F^\times$ but that $\langle a, b\rangle$ is always degenerate.

(b) Prove that $[a, b]$ is nondegenerate for any $a, b \in F$.

(c) Prove that any nondegenerate quadratic form of dimension 2 over F is isometric to a binary quadratic form $[a, b]$ for some $a, b \in F$.

(d) Prove that $\mathbb{H} \cong [0, 0] \cong [0, a]$ for any $a \in F$.

(e) Let $\wp : F \to F$ be the Artin–Schreier map $x \mapsto x^2 + x$. For $a \in F$ prove that $[1, a]$ is isotropic if and only if $a \in \wp(F)$. The group $F/\wp(F)$ plays the role of the group of square classes.

(f) Prove that a nondegenerate quadratic form of dimension 2 over F is isometric to $\mathbb{H}$ if and only if it is isotropic.

(g) Let q be a nondegenerate quadratic form over F. Prove that q is isotropic if and only if $q \cong \mathbb{H} \perp q'$.

(h) Prove that any nondegenerate quadratic form over F can be written as an orthogonal sum

$$\mathop{\perp}_{i=1}^{m} [a_i, b_i] \qquad \text{or} \qquad \langle c\rangle \perp \mathop{\perp}_{i=1}^{m} [a_i, b_i]$$

depending on whether the dimension is even or odd. This is "diagonalization" in characteristic 2.

3. *Prime ideals in the Witt ring.* Let F be a field of characteristic not 2 and P a prime ideal of the Witt ring $W(F)$. Prove the following.

(a) If $W(F)/P$ has characteristic zero, then $W(F)/P \cong \mathbb{Z}$.

(b) If $W(F)/P$ has characteristic $p > 0$, then $W(F)/P \cong \mathbb{Z}/p\mathbb{Z}$.

(c) If $W(F)/P$ has characteristic 2, then $P = I$ is the fundamental ideal.

(d) If P is not the fundamental ideal, then $\{a \in F \mid \langle a \rangle \equiv 1 \pmod{P}\}$ is the set of positive elements of an ordering on F.

Hint. For any $a \in F^\times$, prove the relation $(\langle a \rangle + 1)(\langle a \rangle - 1) = 0$ in $W(F)$.