Yale University Department of Mathematics
Math 608 Introduction to Arithmetic Geometry
Spring 2016

Final Exam (due 5:30 pm Wednesday May 11th)

Guidelines. You may use any external sources, but work by yourself.

Notations. Recall that projective space $\mathbb{P}^n$ parameterizes lines through the origin in an $(n+1)$-dimensional vector space. For any field k, we can view $\mathbb{P}^n(k)$ as the set of tuples $(a_0, \ldots, a_n) \in k^{n+1}$ modulo the diagonal scalar action of $k^\times$. If K/k is a Galois extension with group G, then G acts on $\mathbb{P}^n(K)$ via its natural action on K^{n+1}.

1. Let k be a field of characteristic 2 and $a \in k$ and $b \in k^\times$. Define the generalized quaternion algebra $[a, b)$ by the presentation

$$[a, b) = \langle i, j \,|\, i^2 + i = a,\ j^2 = b,\ ij = ji + j \rangle.$$

Then $[a, b)$ is an associative k-algebra of dimension 4. Prove that the following are equivalent:

(a) $[a, b) \cong M_2(k)$

(b) $[a, b)$ is not a division algebra.

(c) The element b is a norm from the Artin–Schreier extension $k(\alpha)/k$, where α is a root of the equation $x^2 + x = a$.

(d) The projective conic defined by $ax^2 + by^2 = z^2 + zx$ in $\mathbb{P}^2$ has a k-rational point.

2. Let K/k be a finite Galois extension with group G and let $B(K) \subset \mathrm{GL}_2(K)$ be the subgroup of upper triangular matrices.

(a) Identify the quotient $\mathrm{GL}_2(K)/B(K)$ as a G-set with the set $\mathbb{P}^1(K)$.

(b) Show that $H^1(G, B(K))$ is trivial.
Hint. Use the "long" exact sequence in nonabelian Galois cohomology.

(c) Show that $H^1(G, K^+)$ is trivial, here K^+ is the additive group of K.
Hint. Show that $a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ is a G-equivariant representation $K^+ \to B(K)$.

3. Let G be a finite cyclic group of order n and fix a generator σ. Let A be a G-module (i.e., abelian group with G-action). Consider the maps $N : A \to A$ and $\sigma - 1 : A \to A$ defined by

$$N(x) = \sum_{i=0}^{n-1} \sigma^i(x) \qquad \text{and} \qquad (\sigma - 1)(x) = \sigma(x) - x.$$

(a) Verify that the $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a free resolution

$$\cdots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is the usual **augmentation** map or **counit** sending every group element to 1.

(b) Show that this resolution gives the following periodicity on the level of cohomology

$$H^0(G, A) = A^G \qquad \text{and} \qquad H^i(G, A) = \begin{cases} {}_N A/(\sigma - 1)A & \text{if } i \text{ is odd} \\ A^G/NA & \text{if } i \text{ is even} \end{cases}$$

for $i > 0$, where ${}_N A = \ker(N : A \to A)$.

(c) Give formulas for $H^1(G, A)$ when G acts trivially on A.

(d) If K/k is a finite cyclic Galois extension with group G, show that $H^1(G, K^\times) = 1$ gives the original form of Hilbert's Theorem 90: that $x \in K^\times$ satisfies $N_{K/k}(x) = 1$ if and only if $x = \sigma(y)/y$ for some $y \in K^\times$.

4. Let k be a field of characteristic 0 such that $\mathrm{Gal}(\overline{k}/k) \cong \mathbb{Z}/p\mathbb{Z}$ for a prime number p.

(a) Prove that $\mathrm{Br}(k) \cong k^\times / N_{\overline{k}/k}(\overline{k}^\times)$.
Hint. Use the cohomology of cyclic groups in degree 2.

(b) Also, prove that $\mathrm{Br}(k) \cong \mathrm{Br}(k)/p\mathrm{Br}(k) \cong k^\times / k^{\times p}$.
Hint. Use the Kummer sequence $1 \to \mu_p \to \overline{k}^\times \to \overline{k}^\times \to 1$ and its associated long exact sequence in Galois cohomology going up to degree 3, Hilbert's Theorem 90, and the cohomology of cyclic groups.

(c) Conclude that $N_{\overline{k}/k}(\overline{k}^\times) = k^{\times p}$ hence the only possibility is $p = 2$ and $\overline{k} = k(\sqrt{-1})$.
Hint. Show that k contains a primitive pth root of unity (if not try adjoining it), hence that the cyclic extension $\overline{k}/k$ is a Kummer extension, i.e., $\overline{k} = k(\alpha)$ where $\alpha^p = y$ for some $y \in k^\times \smallsetminus k^{\times p}$.

(d) Show that declaring the squares to be positive will equip k with an ordered field structure.

(e) (Artin–Schreier) Prove that if k is a field of characteristic 0 whose absolute Galois group is a nontrivial finite group, then $\overline{k} = k(\sqrt{-1})$ and k is an ordered field where the squares are positive.
Hint. Take a p-Sylow subgroup of the Galois group and use the fact that p-groups are solvable, then iteratively apply the previous results.

Remark. In fact, Artin and Schreier proved that in positive characteristic, the absolute Galois group is either trivial or infinite.