Dartmouth College Department of Mathematics
**Math 71 Algebra**
Fall 2021

Problem Set # 2 (due via Canvas upload by 5 pm, Wednesday, October 13)

**Notation:** $Z_n$ is an abstract cyclic group written multiplicatively.

**Reading:** DF 1.4, 1.6, 2.1, 2.3.

**Problems:**

**1.** DF 1.6 Exercises 2*, 3, 4*, 6*, 7, 9* (here $D_{24}$ is the dihedral group with 24 elements), 14*, 16, 17* (prove that it's always a bijection), 18, 24*, 25.

**2.** DF 2.1 Exercises 2, 6*, 7, 8, 9*, 10, 12, 14.

**3.** DF 2.3 Exercises 2, 5, 8*, 10, 11, 20, 21*, 22*, 23* (Hint: What does 22 tell you about the order of 5 in $(\mathbb{Z}/2^n\mathbb{Z})^\times$?), 25, 26*.

**4.** *Fields of order 4.*

(a) Let $F = \{0, 1, x, y\}$. Prove that there are operations $+$ and $\cdot$ on $F$, such that $1 + x = y$ and $x^2 = y$, making $F$ into a field. (Note that the four elements of $F$ are distinct!) Essentially the problem is to fill out the addition and multiplication tables:

| $+$ | 0 | 1 | $x$ | $y$ |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |
| $x$ | | | | |
| $y$ | | | | |

| $\cdot$ | 0 | 1 | $x$ | $y$ |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |
| $x$ | | | | |
| $y$ | | | | |

You already know certain rows and columns by properties of 0 and 1 in a field!

(b) Let $F_1$ and $F_2$ be fields. A map $\phi : F_1 \to F_2$ is an **isomorphism of fields** if $\phi$ is a bijection satisfying $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ and $\phi(1_{F_1}) = 1_{F_2}$. An isomorphism between a field and itself is called an **automorphism**. Find a non-identity automorphism of the field $F$ of order 4 described above.

(c) Let $F'$ be any field with 4 elements. Prove that there exists an isomorphism $\phi : F \to F'$, where $F$ is the field described above.

This shows that there is a unique "isomorphism class" of field of order 4, which we call $\mathbb{F}_4$.

Dartmouth College, Department of Mathematics, Kemeny Hall, Hanover, NH 03755
*E-mail address*: asher.auel@dartmouth.edu