Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2021

Problem Set # 4 (due via Canvas upload by 5 pm, Wednesday, November 3rd)

Reading: DF 4.5, 5.1–5.2.

Problems: (No textbook problems are required, but they are good practice/interesting/fun.)

0. DF 4.5 Exercises 4–8, 18, 22, 26, 30, 39, 40.
DF 5.1 Exercises 4, 14, 15–18.
DF 5.2 Exercises 2, 3, 5, 6, 7, 8, 9, 11, 14. The notions of **exponent** and **rank** are defined just above the exercise section; the book uses the term **free rank** for what we called the rank of an abelian group.

1. *Solvable up to sixty!* Recall that A_5, which has order 60, is simple. The goal is to:

($\aleph$) Prove that all groups of order < 60 are solvable.

As explain in class, this has two steps. First, use Jordan–Hölder to:

(α) Prove that if 60 is the first order of a finite nonabelian simple group, then all groups of order < 60 are solvable.

Second, prove that every group of order < 60 is not simple. For this, as the abelian simple groups are precisely those of prime order, for each *composite* order $n < 60$, we will try to prove that no group of order n is simple. For example, we already know that no group of order p^α, with $\alpha > 1$, is simple and that no group of order pq, with p and q primes, is simple. Prove the following additional criteria on the order of a group for the group to not be simple:

(a) If G is a finite group of order $p^k m$, with $p \nmid m$ and $m < p$ (more generally, no divisor of m other than 1 is congruent to 1 modulo p), then G has a normal Sylow p-subgroup.

(b) If G is a finite group of order pqr, where p, q, and r are primes with $p < q < r$, then G has a normal Sylow subgroup for at least one of p, q, or r.

(c) If G is a finite group of order $2^k \cdot 3$, with $k \geq 1$, then G is not simple.

(d) If G is a finite group of order $2^k \cdot 5$, with $k \geq 1$, then G is not simple.

(e) If G is a finite group of order $2^2 \cdot 3^k$, with $k \geq 1$, then G is not simple. For $k = 1$, use part (c).

(f) If G is a finite group of order $3^k \cdot 5$, with $k \geq 1$, then G is not simple.

(g) No group of order 56 is simple.

Hints. Part (a) follows from a direct application of the congruence conditions in the Sylow theorems. For (b), assume the contrary and consider the possible number of Sylow r-subgroups, then use this to count the number of elements of order r (any two Sylow r-subgroups intersect only at the identity), combine this with the number of elements of order p and q to find more elements than the order of the group. For (c) and (d), handle k small using the Sylow congruence conditions and then for k large, consider the permutation representation associated to the conjugation action of G on the set of Sylow 2-subgroups. For (e) and (f), do the same using the Sylow 3-subgroups. For (g), if neither the Sylow 2- nor 7-subgroups are normal, start counting elements in these subgroups to reach a contradiction (while any two Sylow 7-subgroups only intersect at the identity, how could Sylow 2-subgroups intersect?).

Finally, use all the criteria you know to handle every composite order < 60. Have fun!

2. *Some isomorphisms.*

(a) For any field F, prove that the center of $\mathrm{GL}_2(F)$ consists of $F^\times$ multiples of the identity matrix. What is the center of $\mathrm{SL}_2(F)$? We denote by $\mathrm{PGL}_2(F) = \mathrm{GL}_2(F)/Z(\mathrm{GL}_2(F))$ and $\mathrm{PSL}_2(F) = \mathrm{SL}_2(F)/Z(\mathrm{SL}_2(F))$.

(b) Prove that $\mathrm{GL}_2(F)$ acts on the set P of lines in F^2 through the origin and that the kernel of this action is the center of $\mathrm{GL}_2(F)$. Here, "line through the origin" is a colloquial term for "1-dimensional subspace." Conclude that $\mathrm{PGL}_2(F)$ acts faithfully on the set P, hence the permutation representation is an injective homomorphism $\mathrm{PGL}_2(F) \to S_P$ to the symmetric group on the elements of P.

(c) Calculate $|\mathrm{PGL}_2(\mathbb{F}_p)|$.

(d) Prove that $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$. (Hint: How many lines through the origin are there in $\mathbb{F}_3^2$?)

(e) Under the isomorphism $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ from the previous part, find all elements of $\mathrm{PGL}_2(\mathbb{F}_3)$ corresponding to 3-cycles.

(f) For an odd prime p, prove that the map $\mathrm{PSL}_2(\mathbb{F}_p) \to \mathrm{PGL}_2(\mathbb{F}_p)$, taking the coset represented by a matrix M to the coset represented by M, is a well defined injective homomorphism whose image has index 2. Notice that for $p = 3$ this is particularly clear!

(g) Prove that the determinant yields a well-defined homomorphism $\det : \mathrm{PGL}_2(\mathbb{F}_3) \to \mathbb{F}_3^\times$. Show that $\mathrm{PSL}_2(\mathbb{F}_3) = \ker(\det)$.

(h) Show that $\mathrm{PSL}_2(\mathbb{F}_3)$ is isomorphic to the subgroup of S_4 generated by the 3-cycles.

(i) Prove that $A_4 \leqslant S_4$ is generated by 3-cycles and conclude that $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4$.

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