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How do I prove that every group of order 4 is simple?

To prove that every group of order 4 is simple, we can use the following approach. A group is simple if it has no nontrivial normal subgroups, meaning it only has two trivial normal subgroups, the identity and the group itself.

Let G be a group of order 4. There are several cases to consider based on the possible group structures of G:

Case 1: G is isomorphic to Z4 (the cyclic group of order 4):

In this case, G is generated by a single element a such that a^4 = e (the identity element). Since G is cyclic, the subgroup generated by a is {e, a, a^2, a^3}. Any proper subgroup of G would have order less than 4, and therefore, it cannot be normal in G. Thus, G is simple.

Case 2: G is isomorphic to Z2 × Z2 (the direct product of two cyclic groups of order 2):

In this case, G has elements (a, b), where a and b are generators of the two cyclic subgroups of order 2. The elements of G are {(e, e), (a, e), (e, b), (a, b)}, where (e, e) is the identity element.

Now, consider a subgroup H of G. H must contain the identity element (e, e) since it is a group. If H contains any of the elements (a, e) or (e, b), then it will also contain their inverses, which are (a^3, e) and (e, b^3), respectively. This is because in Z2 × Z2, every element has order 2, and the only non-identity element in a cyclic group of order 2 is its own inverse. So, H must contain at least three elements: (e, e), (a, e), and (e, b).

Now, if H contains the element (a, b), then it must also contain its inverse, which is (a^3, b^3). However, the subgroup generated by (a^3, b^3) would also contain (e, e), (a, e), and (e, b) since these are the products of (a, b) with (a^3, b^3). Therefore, H would be the entire group G.

Since any subgroup H of G must either be the trivial subgroup {e, e} or the entire group G, there are no proper nontrivial subgroups of G that are normal. Thus, G is simple.

In both cases (Z4 and Z2 × Z2), we have shown that any group of order 4 is simple.

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Is every group of order 4 simple?

No, not every group of order 4 is simple. There are non-simple groups of order 4. The only two groups of order 4 are the cyclic group of order 4 (denoted as C4) and the Klein four-group (denoted as V4 or K4).

1. Cyclic Group of Order 4 (C4):
   - C4 = {e, a, a^2, a^3} where a^4 = e.
   - C4 is a simple group because it has no nontrivial proper subgroups that are normal.
2. Klein Four-Group (V4 or K4):
   - V4 = {e, a, b, c} where a^2 = b^2 = c^2 = e, and ab = ba, ac = ca, bc = cb.
   - V4 is not a simple group. It has three nontrivial proper subgroups: {e, a}, {e, b}, and {e, c}. All of these subgroups are normal in V4, and none of them are simple themselves.

So, while C4 is a simple group of order 4, the Klein four-group V4 is an example of a non-simple group of order 4. Therefore, not every group of order 4 is simple.