Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2023

Problem Set # 2 (due via Canvas upload by 5 pm, Wednesday, October 11)

Notation: $Z_n = \langle x \mid x^n = e \rangle$ is an abstract cyclic group written multiplicatively.

Given groups A and B, their **direct product** is the Cartesian product set $A \times B$, i.e., the set of ordered pairs (a, b) with $a \in A$ and $b \in B$, together with the operation $(a, b) \cdot (a', b') = (aa', bb')$. Then $A \times B$ is a group with identity (e_A, e_B) and inverses $(a, b)^{-1} = (a^{-1}, b^{-1})$.

Reading: DF 1.4, 1.6, 2.1, 2.3, 3.1.

Problems:

1. *Klein four.* Define the **Klein four group** to the group with presentation

$$V_4 = \langle a, b \mid a^2 = b^2 = e, ab = ba \rangle$$

(a) Prove that V_4 has 4 elements, and that each nonidentity element has order 2.

(b) Prove that V_4 is isomorphic to $(\mathbb{Z}/8\mathbb{Z})^\times$, as well as isomorphic to $Z_2 \times Z_2$.

(c) Find every subgroup of S_4 isomorphic to V_4, and determine which are normal subgroups.

2. DF 1.6 Exercises 16*, 17 (prove that it's always a bijection), 18, 20*, 21, 24, 25.

3. DF 1.7 Exercises 12* (what is D_{2n} modulo the kernel of this action?), 20* (write down this subgroup of S_4, see DF 1.2 Exercise 9).

4. DF 2.1 Exercises 2, 6, 7, 8, 9* (determine the order of $\mathrm{SL}_2(\mathbb{F}_p)$), 10, 12, 14.

5. DF 2.2 Exercises 4, 6* 7.

6. DF 2.3 Exercises 5, 8*, 10, 11, 20, 21*, 22, 23*, 26*.

7. *Fields of order 4.*

(a) Let $F = \{0, 1, x, y\}$. Prove that there are operations $+$ and $\cdot$ on F, such that $1 + x = y$ and $x^2 = y$, making F into a field. (Note that the four elements of F are distinct!) Essentially the problem is to fill out the addition and multiplication tables:

$+$	0	1	x	y
0				
1				
x				
y				

$\cdot$	0	1	x	y
0				
1				
x				
y				

You already know certain rows and columns by properties of 0 and 1 in a field!

(b) Let F_1 and F_2 be fields. A map $\phi : F_1 \to F_2$ is an **isomorphism of fields** if ϕ is a bijection satisfying $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ and $\phi(1_{F_1}) = 1_{F_2}$. An isomorphism between a field and itself is called an **automorphism**. Find a non-identity automorphism of the field F of order 4 described above.

(c) Let F' be any field with 4 elements. Prove that there exists an isomorphism $\phi : F \to F'$, where F is the field described above.

This shows that there is a unique "isomorphism class" of field of order 4, which we call $\mathbb{F}_4$.

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