Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2023

Problem Set # 4 (due via Canvas upload by 5 pm, Wednesday, November 1)

Reading: DF 4.5, 5.1–5.2.

Problems: (No textbook problems are required, but they are good practice/interesting/fun.)

0. DF 4.5 Exercises 4–8, 18, 22, 26, 30, 39, 40.
DF 5.1 Exercises 4, 14, 15–18.
DF 5.2 Exercises 2, 3, 5, 6, 7, 8, 9, 11, 14. The notions of **exponent** and **rank** are defined just above the exercise section; the book uses the term **free rank** for what we called the rank of an abelian group.

1. *Solvable up to sixty!* Recall that A_5, which has order 60, is simple. The goal is to:

($\aleph$) Prove that all groups of order < 60 are solvable.

As explain in class, this has two steps. First, use Jordan–Hölder to:

(α) Prove that if 60 is the first order of a finite nonabelian simple group, then all groups of order < 60 are solvable.

Second, prove that every nonabelian group of order < 60 is not simple. For this, as the abelian simple groups are precisely those of prime order, for each *composite* order $n < 60$, we will try to prove that any group of order n is not simple. For example, we already know that no group of order p^α, with $\alpha > 1$, is simple and that no group of order pq, with p and q primes, is simple. Prove the following additional criteria on the order of a group for the group to not be simple:

(a) If G is a finite group of order $p^k m$, with $p \nmid m$ and $m < p$ (more generally, no divisor of m other than 1 is congruent to 1 modulo p), then G has a normal Sylow p-subgroup.

(b) If G is a finite group of order pqr, where p, q, and r are primes with $p < q < r$, then G has a normal Sylow subgroup for at least one of p, q, or r.

(c) If G is a finite group of order $2^k \cdot 3$, with $k \geq 1$, then G is not simple.

(d) If G is a finite group of order $2^k \cdot 5$, with $k \geq 1$, then G is not simple.

(e) If G is a finite group of order $2^2 \cdot 3^k$, with $k \geq 1$, then G is not simple. For $k = 1$, use part (c).

(f) If G is a finite group of order $3^k \cdot 5$, with $k \geq 1$, then G is not simple.

(g) No group of order 56 is simple.

Hints. Part (a) follows from a direct application of the congruence conditions in the Sylow theorems. For (b), assume the contrary and consider the possible number of Sylow r-subgroups, then use this to count the number of elements of order r (any two Sylow r-subgroups intersect only at the identity), combine this with the number of elements of order p and q to find more elements than the order of the group. For (c) and (d), handle k small using the Sylow congruence conditions and then for k large, consider the permutation representation associated to the conjugation action of G on the set of Sylow 2-subgroups. For (e) and (f), do the same using the Sylow 3-subgroups. For (g), if neither the Sylow 2- nor 7-subgroups are normal, start counting elements in these subgroups to reach a contradiction (while any two Sylow 7-subgroups only intersect at the identity, how could Sylow 2-subgroups intersect?).

Finally, use all the criteria you know to handle every composite order < 60. Have fun! how much higher can you go using these same tools?

(h) (Optional.) What is the largest number $N > 60$ you can find such that no group of composite order n, with $61 \leq n \leq N$, is simple?

2. *GAI Engagement.* Ask `Fran` to prove a correct mathematical statement in algebra, then convince `Fran` that the original statement you asked about is actually false (you may have to be persistent), and see how `Fran` tries to fix their proof. Include a screenshot or print-out of the conversation.