Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2023

Midterm 1 Review Sheet

Directions: The first midterm exam will be handed out, in physical paper form, at the end of class on Tuesday, October 3rd and will be picked up at the beginning of class on Thursday, October 5th. The exam is intended to take 2 hours, however, you can spend a maximum of 4 hours, in any single continuous stretch, working on the midterm exam, and you must write your starting time and ending time at the top of your exam paper. No electronic devices, notes, external resources, nor textbooks will be allowed to be used while you are working on the exam. On all problems, you will need to write your thoughts/proofs in a coherent way to get full credit.

Topics covered and practice problems:

- Basic set theory and functions (injections, surjections, bijections). Definition of a group. Presentation with generators and relations. Modular arithmetic. Multiplicative group modulo n.
  DF 0.1 Exercises 4–6; DF 0.2 Exercises 7, 8, 11; DF 0.3 Exercises 4, 5, 7, 9; DF 1.1 Exercises 1, 2, 5–9, 12–14, 21, 28, 31, 36.

- Dihedral groups. Symmetric groups. Disjoint cycle decomposition of permutations. Matrix groups over a field.
  DF 1.2 Exercises 4–6, 9; DF 1.3 Exercises 9–19; DF 1.4 Exercises 7, 10.

- Homomorphisms and isomorphisms. Kernel. Image.
  DF 1.6 Exercises 3–9, 11, 15–16, 19, 21–22, 25.

- Group actions. Permutation representation. Kernel. Faithful. Transitive. Orbit. Stabilizer. Left multiplication action. Conjugation action. Conjugacy classes. Cycle type and conjugacy in the symmetric group. Centralizer.
  DF 1.7 Exercises 1–3, 5–6, 8–13, 20, 21, 23; DF 4.1 Exercises 1–6; DF 4.2 Exercises 1–3. DF 4.3 Exercises 2–3, 7, 10–12.

- Subgroups. Cyclic subgroups.
  DF 2.1 Exercises 1–5, 14; DF 2.3 Exercises 1–5, 10–14.

Practice exam questions (not representative of the length of the exam):

1. There will be several True/False problems covering a range of topics so far: isomorphic groups, orders of elements, homomorphisms, Lagrange's theorem, and group actions.

2. Let $V_1 \subset \mathbb{R}^2$ be the subset of all vectors whose slope is an integer. Let $V_2 \subset \mathbb{R}^2$ be the subset of all vectors whose slope is a rational number. Determine if V_1 and/or V_2 is a subgroup of $\mathbb{R}^2$, with usual vector addition.

3. Write down a nontrivial homomorphism $\mathbb{Z}/36\mathbb{Z} \to \mathbb{Z}/48\mathbb{Z}$ and compute its image and kernel. Can you find an injective homomorphism or a surjective homomorphism?

4. How many elements of order 6 are there in $\mathbb{Z}/5\mathbb{Z}$, S_6, and $(\mathbb{Z}/7\mathbb{Z})^\times$?

5. Prove that $11^{104} + 1$ is divisible by 17.

6. Write down two elements of S_{10} that generate a subgroup isomorphic to D_{10}.

7. Choose an ordering of the six elements in S_3, then compute how each element is sent under the permutation representation $S_3 \to S_6$ induced by both the left multiplication action and the conjugation action of S_3.

8. Consider the permutation representation $S_n \to S_{n!}$ associated to the left multiplication action on S_n. Describe the cycle type in $S_{n!}$ of the image of an n-cycle in S_n.

9. Prove that the centralizer $C_{S_n}\big((12)(34)\big)$ has $8\,(n-4)!$ elements for $n \geq 4$ and explicitly determine all of them.

10. Consider the action of S_5 on the 10 subsets of $\{1,2,3,4,5\}$ of order 2. Show that this action is transitive. Write down the stabilizer of $\{4,5\}$ explicitly as a subgroup of S_5 and then determine its isomorphism type (first start by computing its order).

11. Show that the set of nonzero matrices of the form

$$\begin{pmatrix} a & 3b \\ b & a \end{pmatrix}$$

is a cyclic subgroup of $\mathrm{GL}_2(\mathbb{F}_5)$. What is the order of this subgroup?

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