Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2023

Midterm 2 Review Sheet

Directions: The second midterm exam will be handed out, in physical paper form, at the end of class on Tuesday, October 24rd and will be picked up at the beginning of class on Thursday, October 26th. While it's intended to take 2 hours, you can spend a maximum of 4 hours, in any single continuous stretch, working on the midterm exam, and you must write your starting time and ending time in the appropriate place. No electronic devices, notes, external resources, nor textbooks will be allowed to be used while you are working on the exam. On all problems, you will need to write your thoughts/arguments in a coherent way to get full credit.

New topics covered (since the last midterm) and practice DF problems:

- Group actions. Orbit-stabilizer theorem. Left multiplication action. Conjugation action. Conjugacy classes. Center. Class equation. Conjugacy classes in S_n and cycle type. p-groups.
  DF 1.7 Exercises 1–3, 5–6, 8–13, 20, 21, 23; DF 4.1 Exercises 1–6; DF 4.2 Exercises 1–3. DF 4.3 Exercises 2–3, 5, 7, 10–12, 29, 31–32.
- Subgroups. Cyclic subgroups. Centralizers. Generators. Lattice of subgroups.
  DF 2.1 Exercises 1–5, 14; DF 2.2 Exercises 1–2, 7; DF 2.3 Exercises 1–5, 10–14. DF 2.4 Exercises 6–9; DF 2.5 Exercises 4, 9–10, 15.
- Quotient groups. Cosets. Isomorphism theorems.
  DF 3.1 Exercises 6–13, 17, 20–21, 33–35; DF 3.2 Exercises 8, 13–17, 21–23; DF 3.3 Exercises 1, 8;
- Composition series. Simple groups. Simplicity of A_5.
  DF 3.4 Exercises 1–2 (just do D_8); DF 4.3 Exercises 20–22.
- Alternating groups.
  DF 3.5 Exercises 1–4, 6, 8, 9, 10, 12.

Practice exam questions (not representative of the length of the exam):

1. There will be several True/False problems covering a range of topics so far: isomorphic groups, orders of elements, homomorphisms, Lagrange's theorem, conjugacy classes, and group actions.

2. Make yourself familiar with the lattice of subgroups and conjugacy classes of some common groups, including $\mathbb{Z}/n\mathbb{Z}$, S_3, D_8, A_4. Know all the normal subgroups and what the quotients are.

3. How many elements of order 6 are there in A_5 and A_6?

4. Let $V_4 \subset S_4$ be the subset consisting of $(2,2)$-cycles and the identity. Prove that V_4 is a normal subgroup and that the quotient S_4/V_4 is isomorphic to S_3.

5. Prove that if $\sigma = (12m) \in S_n$, then $C_{S_n}(\sigma)$ is generated by σ and all those permutations fixing $1, \ldots, m$.

6. Find the highest power of p dividing the order of $\mathrm{GL}_n(\mathbb{F}_p)$. Find a subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ of that order. (Hint: Think triangularly.)