Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2025

Problem Set # 3 (due via Canvas upload by 5 pm, Wednesday, October 15)

Reading: DF 2.2–2.5, 3.1–3.3.

Problems:

1. DF 2.2 Exercises 4, 6* 7.

2. DF 2.3 Exercises 5, 8*, 10, 11, 20, 21*, 22, 23*, 26*.

3. DF 2.4 Exercises 2, 5, 6* (show it's isomorphic to the Klein four group), 7*, 8, 11, 12, 14*, 19

4. DF 2.5 Exercises 4, 10, 12*, 14*, 15.

5. Prove Euler's Theorem: If a and n are relatively prime integers, then $a^{\varphi(n)} \equiv 1 \pmod{n}$.
Hint. Use Lagrange's theorem on the group $(\mathbb{Z}/n\mathbb{Z})^\times$.

6. Show that for all $n, m \geq 1$, the group S_{n+m} contains a subgroup isomorphic to $S_n \times S_m$. Conclude that $n!m!$ divides $(n+m)!$.

7. *Tricks with Euler's theorem.* You can only use pencil and paper!

(a) Prove that every element of $(\mathbb{Z}/72\mathbb{Z})^\times$ has order dividing 12. (Hint: This is better than what a straight application of Euler's theorem will give you! Try applying Euler's theorem to a pair of relatively prime divisors of 72.)

(b) Prove that if n is a positive integer, then n and n^5 have the same last digit. Now Google "Fifth root trick" and watch the Numberphile video.

(c) Find the last two digits of the huge number $3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}$ where there are 2025 threes appearing! (Hint: Do nested applications of Euler's theorem.)

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