Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2025

Problem Set # 6 (due via Canvas upload by 5 pm, Monday, November 17)

Remark: This problem set is *extra credit* and can be used to recover up to 5% of your grade. There are some challenging problems below, so have fun!

Problems:

1. *Final DF exercises!*
DF 7.4 Exercises 8, 13, 33 (see Example 4 on page 255), 34.
DF 7.6 Exercises 1*, 3*, 4, 5, 7.
DF 8.1 Exercises 11a, 12*.
DF 8.2 Exercises 2, 3, 5*.
DF 9.1 Exercises 4, 5*, 6, 7, 13.

2. Call a positive integer n *special* if there exists an integer m with $1 < m < n$ so that

$$1 + 2 + \cdots + (m-1) = (m+1) + \cdots + n.$$

For example, $n = 8$ is special with $m = 6$, while $n = 7$ is not special. Find all positive integers that are special. **Hint.** Relate the pairs (n, m) to integer solutions of $a^2 - 2b^2 = 1$, i.e. units in $\mathbb{Z}[\sqrt{2}]$ of norm 1. You can use the book's description of the set of units in $\mathbb{Z}[\sqrt{2}]$.

3. Prove the following presentations.
(a) $A_4 = \langle x, y \mid x^2 = y^3 = (xy)^3 = 1 \rangle$
(b) $S_4 = \langle x, y \mid x^2 = y^3 = (xy)^4 = 1 \rangle$
(c) $A_5 = \langle x, y \mid x^2 = y^3 = (xy)^5 = 1 \rangle$

4. For a field F, denote by $\mathrm{PGL}_2(F)$ and $\mathrm{PSL}_2(F)$ the quotients of $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ by their respective normal subgroups consisting of nonzero scalar multiples of the identity.
(a) Prove that for any field F, the kernel of the action of $\mathrm{GL}_2(F)$ on the lines through the origin in the vector space F^2 consists of the subgroup of nonzero scalar matrices. Deduce that the induced action of $\mathrm{PGL}_2(F)$ on the set of lines is faithful.
(b) Prove that for a finite field $\mathbb{F}_q$ with q elements, the number of lines through the origin in the vector space $\mathbb{F}_q^2$ is $q + 1$.
(c) Recall, from Problem Set # 2, the construction of the field $\mathbb{F}_4$ of order 4. Prove that $\mathrm{PGL}_2(\mathbb{F}_4) \cong \mathrm{PSL}_2(\mathbb{F}_4) \cong \mathrm{SL}_2(\mathbb{F}_4) \cong A_5$. **Hint.** Consider the action on the set of lines through the origin in the vector space $\mathbb{F}_4^2$.
(d) Prove that $\mathrm{PGL}_2(\mathbb{F}_5) \cong S_5$ and that $\mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$. **Hint.** The action on the set of lines through the origin in the vector space $\mathbb{F}_5^2$ gives an injective homomorphism $\mathrm{PGL}_2(\mathbb{F}_5) \to S_6$. Count the number of $(2,2,2)$-cycles in S_6 that are not in the image of this homomorphism, then let $\mathrm{PGL}_2(\mathbb{F}_5)$ act on them by conjugation to determine a new permutation homomorphism.

5. *Some linear algebra over the field of order 9.*
(a) Prove that $\mathbb{F}_3[i] = \{0, \pm 1, \pm i, \pm 1 \pm i\}$, where $i^2 = -1$ and all other arithmetic is done modulo 3, is a field of order 9, which we call $\mathbb{F}_9$.
(b) Prove that $\mathrm{PGL}_2(\mathbb{F}_9)$ is *not* isomorphic to S_6, even though they have the same order. **Hint.** Use linear algebra to bound the size of some of the conjugacy classes in $\mathrm{PGL}_2(\mathbb{F}_9)$ and compare with what we know about S_6.
(c) Prove, on the other hand, that $\mathrm{PSL}_2(\mathbb{F}_9) \cong A_6$. **Hint.** Find a subgroup of $\mathrm{PSL}_2(\mathbb{F}_9)$ isomorphic to A_5, and then act on the 6 cosets to get a permutation representation.