Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2025

Midterm Review Sheet

Directions: The midterm exam will be conducted in-class during the X-hour on Friday, October 24, 3:30–4:20 pm. No electronic devices, notes, external resources, nor textbooks will be allowed to be used while you are working on the exam. However, you are allowed to bring one handwritten 2-sided 3×5 inch index card (or equivalently sized sheet of paper) to refer to during the exam. On all problems, you will need to write your thoughts/proofs in a coherent way to get full credit.

Topics covered and practice problems:

- Basic set theory and functions (injections, surjections, bijections). Definition of a group. Presentation with generators and relations. Modular arithmetic. Multiplicative group modulo n.
  DF 0.1 Exercises 4–6; DF 0.2 Exercises 7, 8, 11; DF 0.3 Exercises 4, 5, 7, 9; DF 1.1 Exercises 1, 2, 5–9, 12–14, 21, 28, 31, 36.

- Dihedral groups. Symmetric groups. Disjoint cycle decomposition of permutations. Matrix groups over a field. Alternating groups.
  DF 1.2 Exercises 4–6, 9; DF 1.3 Exercises 9–19; DF 1.4 Exercises 7, 10. DF 3.5 Exercises 2–6, 9–12, 15–16.

- Homomorphisms and isomorphisms. Kernel. Image.
  DF 1.6 Exercises 3–9, 11, 15–16, 19, 21–22, 25.

- Group actions. Permutation representation. Kernel. Faithful. Transitive. Orbit. Stabilizer. Left multiplication action. Conjugation action. Conjugacy classes. Cycle type and conjugacy in the symmetric group. Centralizer.
  DF 1.7 Exercises 1–3, 5–6, 8–13, 20, 21, 23; DF 4.1 Exercises 1–6; DF 4.2 Exercises 1–3. DF 4.3 Exercises 2–3, 7, 10–12.

- Subgroups. Cyclic subgroups. Centralizers. Normalizers. Generators. Lattice of subgroups.
  DF 2.1 Exercises 1–5, 14; DF 2.2 Exercises 1–2, 7, 10–11; DF 2.3 Exercises 1–5, 10–14. DF 2.4 Exercises 6–9, 18; DF 2.5 Exercises 4, 6–10.

- Quotient groups. Cosets. Isomorphism theorems. Composition series. Simple groups.
  DF 3.1 Exercises 6–13, 31, 33–34; DF 3.2 Exercises 4, 8, 13–16, 22–23; DF 3.3 Exercises 1, 3, 8; DF 3.4 Exercises 1–2.

Practice exam questions (not representative of the length of the exam):

1. There will be several True/False problems covering a range of topics so far: isomorphic groups, orders of elements, homomorphisms, Lagrange's theorem, and group actions.

2. Write down all homomorphisms $\mathbb{Z}/36\mathbb{Z} \to \mathbb{Z}/48\mathbb{Z}$ and compute their image and kernel.

3. Write down all homomorphisms $\mathbb{Z}/18\mathbb{Z} \to D_{24}$ and compute their image and kernel.

4. Write down all homomorphisms $D_{12} \to A_5$ and compute their image and kernel.

5. How many elements of order 15 are there in $\mathbb{Z}/60\mathbb{Z}$, $(\mathbb{Z}/77\mathbb{Z})^\times$, and A_8?

6. Write down two elements of S_{10} that generate a subgroup isomorphic to D_{10}.

7. Choose an ordering of the six elements in S_3, then compute how each element is sent under the permutation representation $S_3 \to S_6$ induced by both the left multiplication action and the conjugation action of S_3.

8. Consider the permutation representation $S_n \to S_{n!}$ associated to the left multiplication action on S_n. Describe the cycle type in $S_{n!}$ of the image of an n-cycle in S_n.

9. Prove that the centralizer $C_{S_n}\big((12)(34)\big)$ has $8\,(n-4)!$ elements for $n \geq 4$ and explicitly determine all of them.

10. Consider the action of S_5 on the 10 subsets of $\{1,2,3,4,5\}$ of order 2. Show that this action is transitive. Write down the stabilizer of $\{4,5\}$ explicitly as a subgroup of S_5 and then determine its isomorphism type (first start by computing its order).

11. Consider the action of D_{12} on the set of triangles inscribed in the regular hexagon (whose vertices are vertices of the hexagon). Describe the orbits of this action and for each orbit, describe the stabilizer of an element in that orbit.

12. Write down all normal subgroups of D_{24} and describe the resulting quotient groups.

13. Consider the action of S_4 on the set of all three $(2,2)$-cycles in S_4. Show that this action is transitive. Write down the stabilizer of $(12)(34)$ explicitly as a subgroup of S_4. What is the kernel K of the resulting permutation homomorphism $\pi : S_4 \to S_3$? Describe the quotient group S_4/K.