Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2025
Quiz 2 Review Sheet

Directions: The second in-class quiz will be Tuesday, November 11, during the last 15 minutes of class. No electronic devices, notes, external resources, nor textbooks will be allowed during the quiz. The quiz will not feature problems that require full proofs to be written. However, on all problems, you will need to briefly explain your thoughts in a coherent way, and which theorems you are appealing to, to get full credit.

Topics covered and practice problems:

- Sylow p-subgroups. Sylow's Theorem. Number of Sylow p-subgroups. Applications to groups of small order: pq, $2^k \cdot 3$, $4 \cdot 3^k$, 56, etc. DF 4.5 Exercises 4–9, 13–28, 39–40, 45, 56.
- Direct products. Fundamental theorem of finitely generated abelian groups. Invariant factors and elementary divisors. DF 5.1 Exercises 1, 4, 10, 14, 18; DF 5.2 Exercises 1–6, 9, 10, 15; DF 5.3 Just Read It.
- Rings. Division rings. Quaternion rings. Quadratic fields. Quadratic integer rings. Polynomial rings. Group rings. Subrings. Zero divisors. Group of units. Integral domains. Finite integral domain is a field. DF 7.1 Exercises 3–6, 8, 12, 13, 14, 17, 24, 25, 29; DF 7.2 Exercises 1–4, 5a, 6, 9–11.
- Ring homomorphisms. Ideals. Quotient rings. Isomorphism theorems for rings. Prime ideals. Maximal ideals. DF 7.3 Exercises 1–14, 18–21, 23, 24–26, 28, 31–32, 34, 36–37; DF 7.4 Exercises 4–9, 11, 13–19, 22, 27.

More practice problems (not representative of the length of the quiz):

1. Find a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$. (Hint: Think upper triangular.)

2. Classify all groups of order 157, 169, and 187 up to isomorphism.

3. Let p, q, r be prime numbers (not necessarily distinct). Depending on the values of p, q, r, determine the number of abelian groups of order $(pqr)^2$.

4. Consider the subset R of $M_2(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Prove that R is a subring of $M_2(\mathbb{R})$. Prove that R is a commutative ring with 1, but is not an integral domain. Find all idempotents in R (an idempotent is an element x such that $x^2 = x$). Find all nilpotent elements in R (a nilpotent element is x such that $x^n = 0$ for some $n \geq 0$). Define $\varphi : R \to \mathbb{R}$ by

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto a - b.$$

Show that φ is a ring homomorphism. Determine the kernel of φ as well as $R/\ker(\varphi)$. Is $\ker(\varphi)$ a prime ideal or a maximal ideal?

5. Find a unit of every degree in $\mathbb{Z}/4\mathbb{Z}[x]$.