

Problem Set # 5 (due via Canvas upload by midnight Friday 9 February)

Notation: Let K and L be subfields of a field M . The **compositum** of K and L , denoted KL , is defined to be the smallest subfield of M containing both K and L , equivalently, the intersection of all subfields of M containing K and L . If additionally K and L are both extensions of a field F , we say that the extensions K/F and L/F are **linearly disjoint** if any F -linearly independent subset of K is L -linearly independent in KL and if any F -linearly independent subset of L is K -linearly independent in KL .

Problems:

- Let F be a field and K/F and L/F be subextensions of a field extension M/F .
 - Prove that if $K = F(\alpha_1, \dots, \alpha_n)$ and $L = F(\beta_1, \dots, \beta_m)$ are finitely generated, then $KL = F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$.
 - Prove that if K/F and L/F are finite then KL/F is finite and $[KL : F] \leq [K : F][L : F]$ with equality if and only if K/F and L/F are linearly disjoint.
Hint. Prove that if x_1, \dots, x_n is an F -basis for K and y_1, \dots, y_m is an F -basis for L , then the products $x_i y_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ span KL/F and are an F -basis if and only if K/F and L/F are linearly disjoint.
 - Prove that finite extensions K/F and L/F of relatively prime degree are linearly disjoint.
 - Prove that if K/F and L/F are linearly disjoint then $K \cap L = F$. Find an example showing that the converse is false.
 - Prove that if K/F and L/F are linearly disjoint finite Galois extensions then the map $\text{Gal}(KL/F) \rightarrow \text{Gal}(K/F) \times \text{Gal}(L/F)$ defined by $\sigma \mapsto (\sigma|_K, \sigma|_L)$ is an isomorphism.
- Let K/F be a finite extension of fields and $f(x) \in F[x]$ an irreducible polynomial of degree n .
 - Prove that if n does not divide $[K : F]$ then $f(x)$ has no roots in K .
 - Prove that if n is relatively prime to $[K : F]$ then $f(x)$ is irreducible over K .
- Let F be a field, $f(x)$ a polynomial of degree n over F with splitting field E/F .
 - Let K/F be a subextension of E/F . Prove that E/K is a splitting field of $f(x)$ considered as a polynomial over K .
 - Prove that $[E : F]$ divides $n!$. (We only had $[E : F] \leq n!$ before.)
Hint. Use induction on n , and deal with cases of f reducible or irreducible separately. At some point you'll need the fact that $a!b!$ divides $(a+b)!$, which you should also prove.
- Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Determine the \mathbb{Q} -automorphism group $\text{Aut}_{\mathbb{Q}}K$ by writing down all the elements as automorphisms and also by describing the isomorphism class of the group.