DARTMOUTH COLLEGE DEPARTMENT OF MATHEMATICS
Math 81/111 Abstract Algebra
Winter 2024

Take Home Midterm Exam (due via Canvas upload my midnight Monday 19 February)

Instructions: While working on this exam, you are not allowed to discuss with anyone (except me), nor consult internet resources, besides the two main textbooks (DF and FT). The idea is to use tools that we have developed so far in lecture and in the problem sets; if you are wondering whether you are allowed to invoke a certain theorem found in the texts, please ask me.

Problems:

1. *Construct it!*

(a) Prove that if n and m are relatively prime positive integers such that a regular n-gon and a regular m-gon is constructible, then a regular nm-gon is constructible.
(In case you know it, you cannot use Gauss's classification of constructible n-gons.)

(b) Prove that any root of $x^{128} + x^{96} + x^{64} + x^{32} + 1$ is constructible.

2. *Simple splitting.* Let p be a prime number.

(a) Prove that the splitting field of $x^p - 3$ has degree $p(p-1)$ over $\mathbb{Q}$.

(b) Find a simple generator for the splitting field of $x^3 - p$ over $\mathbb{Q}$.

3. *Embeddings.* For each triple of fields F, K, and L, calculate the size of $\mathrm{Hom}_F(K, L)$.

(a) $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{2})$, $L = \mathbb{Q}(\sqrt[6]{2})$

(b) $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, $L = \mathbb{Q}(\sqrt{\sqrt{2} + \sqrt{3}})$

(c) $F = \mathbb{Q}$, $K = \mathbb{Q}[x]/(x^3 - x - 1)$, $L = \mathbb{R}$

(d) $F = \mathbb{F}_2$, $K = \mathbb{F}_4$, $L = \mathbb{F}_8$

4. *Prime cyclotomic.* Let p be an odd prime number. Prove that $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is a Galois extension and that there is an isomorphism of groups

$$\begin{aligned}(\mathbb{Z}/p\mathbb{Z})^\times &\to \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \\ j &\mapsto \left(\varphi_j : \zeta_p \mapsto \zeta_p^j\right)\end{aligned}$$

Deduce that the Galois group is cyclic of order $p-1$. Find all subfields $L \subset \mathbb{Q}(\zeta_p)$ such that $[L : \mathbb{Q}] = (p-1)/2$.

5. *Galois dreams of eight.* Let $K/\mathbb{Q}$ be the splitting field of the polynomial $x^4 + 1 \in \mathbb{Q}[x]$. Compute the degree $[K : \mathbb{Q}]$ and the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Draw the lattice of subextensions of $K/\mathbb{Q}$, providing a simple generator for each, and the corresponding lattice of subgroups of G under the Galois correspondence.

6. *Dihedral.* Let F be a field of characteristic $\neq 2$ and $a, b \in F$. Let $f(x) = x^4 + ax^2 + b \in F[x]$. Assume that $f(x)$ is separable. Let K be the splitting field of $f(x)$ over F. Prove that the Galois group of K/F is isomorphic to a subgroup of the dihedral group D_8 of order 8.

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