

DARTMOUTH COLLEGE DEPARTMENT OF MATHEMATICS  
**Math 81/111 Abstract Algebra**  
Winter 2025

Problem Set # 1 (due via Canvas upload by 5 pm on Friday 17 January)

**Reading:** DF 9.1–9.4, FT 1, pp. 7–14.

**Problems:**

1. For  $f(x) = x^4 - 1$  and  $g(x) = 3x^2 + 3x$  find: the quotient and remainder after dividing  $f$  by  $g$ ; the gcd of  $f$  and  $g$ ; and the expression of this gcd in the form  $af + bg$  for some  $a, b \in \mathbb{Q}[x]$ . For the last two, you'll need to recall the Euclidean Algorithm and the Bezout Identity.

2. Prove that two polynomials  $f, g \in \mathbb{Z}[x]$  are relatively prime in  $\mathbb{Q}[x]$  (i.e., they share no common nonconstant factor) if and only if the ideal  $(f, g) \subset \mathbb{Z}[x]$  contains a nonzero integer.

3. Decide whether each of the following polynomials is irreducible, and if not, then find the factorization into monic irreducibles.

(a)  $x^4 + 1 \in \mathbb{R}[x]$

(b)  $x^4 + 1 \in \mathbb{Q}[x]$

(c)  $x^7 + 11x^3 - 33x + 22 \in \mathbb{Q}[x]$

(d)  $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$

(e)  $x^3 - 7x^2 + 3x + 3 \in \mathbb{Q}[x]$

4. *Irreducible polynomials over finite fields.* Let  $\mathbb{F}_3$  be the field with three elements.

(a) Determine all the monic irreducible polynomials of degree  $\leq 3$  in  $\mathbb{F}_3[x]$ .

(b) Determine the number of monic irreducible polynomials of degree 4 in  $\mathbb{F}_3[x]$ .

5. *Symmetric polynomials.* Let  $R$  be a commutative ring with 1 and  $R[x_1, \dots, x_n]$  the ring of polynomials in the variables  $x_1, \dots, x_n$  with coefficients in  $R$ . Consider the symmetric group  $S_n$  acting on the set  $\{x_1, \dots, x_n\}$  by permutations. Extend this action linearly to  $R[x_1, x_2, \dots, x_n]$ ; for example, if  $\sigma = (123) \in S_3$ , then

$$\sigma \cdot (x_1x_2 - 2x_3^2 + 3x_2x_3^2) = x_2x_3 - 2x_1^2 + 3x_3x_1^2.$$

Then this action satisfies  $\sigma \cdot (f + g) = \sigma \cdot f + \sigma \cdot g$  and  $\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g)$  for all  $\sigma \in S_n$  and all  $f, g \in R[x_1, \dots, x_n]$ .

(a) Let  $S \subset R[x_1, \dots, x_n]$  be the subset fixed under the action of  $S_n$ . Prove that  $S$  is a subring with 1. This is called the **ring of symmetric polynomials**.

(b) For each  $n \geq 0$ , define polynomials  $e_i \in R[x_1, \dots, x_n]$  by  $e_0 = 1$  and

$$e_1 = x_1 + \cdots + x_n, \quad e_2 = \sum_{1 \leq i < j \leq n} x_i x_j, \quad \dots, \quad e_n = x_1 \cdots x_n$$

and  $e_k = 0$  for  $k > n$ . In words,  $e_k$  is the sum of all distinct products of subsets of  $k$  distinct variables. Prove that each  $e_k$  is a symmetric polynomial. These are called the **elementary symmetric polynomials**.

(c) The **generic polynomial** of degree  $n$  is the polynomial

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

in the ring  $R[x_1, \dots, x_n][x]$  of polynomials in  $x$  with coefficients in  $R[x_1, \dots, x_n]$ . Prove (by induction) that

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n) = x^n - e_1 x^{n-1} + e_2 x^{n-2} + \cdots + (-1)^n e_n = \sum_{j=0}^n (-1)^{n-j} e_{n-j} x^j.$$

(d) For each  $k \geq 1$ , define the **power sums**  $p_k = x_1^k + \cdots + x_n^k$  in  $R[x_1, \dots, x_n]$ . Clearly, the power sums are symmetric. Verify the following identities by hand:

$$p_1 = e_1, \quad p_2 = e_1 p_1 - 2e_2, \quad p_3 = e_1 p_2 - e_2 p_1 + 3e_3$$

In general **Newton's identities** in  $R[x_1, \dots, x_n]$  are (recall that  $e_k = 0$  for  $k > n$ ):

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k k e_k = 0.$$

Prove Newton's identities whenever  $k \geq n$ .

Hint. For each  $i$ , consider the equation in part (c) for  $f(x_i)$  and sum all these equations together. This gives Newton's identity for  $k = n$ . Set extra variables to zero to get the identities for  $k > n$  from this. (Fun. Can you come up with a proof when  $1 \leq k \leq n$ ?)

6. *Use the force, my Newton!*

(a) If  $x, y, z$  are complex numbers satisfying

$$x + y + z = 1, \quad x^2 + y^2 + z^2 = 2, \quad x^3 + y^3 + z^3 = 3,$$

then prove that  $x^n + y^n + z^n$  is rational for any positive integer  $n$ .

(b) Calculate  $x^4 + y^4 + z^4$ .

(c) Prove that each of  $x, y, z$  are not rational numbers.