Dartmouth College Department of Mathematics
Math 81/111 Abstract Algebra
Winter 2025
Problem Set # 5 (due via Canvas upload by midnight Friday 28 February)

Notation: Let K and L be subfields of a field M. The **compositum** of K and L, denoted KL, is defined to be the smallest subfield of M containing both K and L, equivalently, the intersection of all subfields of M containing K and L. If additionally K and L are both extensions of a field F, we say that the extensions K/F and L/F are **linearly disjoint** if any F-linearly independent subset of K is L-linearly independent in KL and if any F-linearly independent subset of L is K-linearly independent in KL.

Problems:

1. Let F be a field and K/F and L/F be subextensions of a field extension M/F.

(a) Prove that if $K = F(\alpha_1, \ldots, \alpha_n)$ and $L = F(\beta_1, \ldots, \beta_m)$ are finitely generated, then $KL = F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$.

(b) Prove that if K/F and L/F are finite then KL/F is finite and $[KL : F] \leq [K : F]\,[L : F]$ with equality if and only if K/F and L/F are linearly disjoint.
Hint. Prove that if $x_1, \ldots, x_n$ is an F-basis for K and $y_1, \ldots, y_m$ is an F-basis for L, then the products $x_i y_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ span KL/F and are an F-basis if and only if K/F and L/F are linearly disjoint.

(c) Prove that finite extensions K/F and L/F of relatively prime degree are linearly disjoint.

(d) Prove that if K/F and L/F are linearly disjoint then $K \cap L = F$. Find an example showing that the converse is false.

(e) Prove that if K/F and L/F are linearly disjoint finite Galois extensions then KL/F is Galois and the map $\mathrm{Gal}(KL/F) \to \mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$ defined by $\sigma \mapsto (\sigma|_K, \sigma|_L)$ is a group isomorphism.

2. Let K/F be a finite extension of fields and $f(x) \in F[x]$ an irreducible polynomial of degree n.

(a) Prove that if n does not divide $[K : F]$ then $f(x)$ has no roots in K.

(b) Prove that if n is relatively prime to $[K : F]$ then $f(x)$ is irreducible over K.

3. Let F be a field, $f(x)$ a polynomial of degree n over F with splitting field E/F.

(a) Let K/F be a subextension of E/F. Prove that E/K is a splitting field of $f(x)$ considered as a polynomial over K.

(b) Prove that $[E : F]$ divides $n!$. (We only had $[E : F] \leq n!$ before.)
Hint. Use induction on n, and deal with cases of f reducible or irreducible separately. At some point you'll need the fact that $a!b!$ divides $(a + b)!$, which you should also prove.

4. This problem will show you a tower of Galois extensions $K/L/F$, with K/F radical but L/F not radical. Let $K = \mathbb{Q}(\zeta_7)$ and $L = \mathbb{Q}(\zeta_7 + \overline{\zeta}_7)$.

(a) Prove that $K/\mathbb{Q}$ is radical.

(b) Prove that $L/\mathbb{Q}$ is not radical. **Warning.** A simple extension $F(\alpha)$ can be radical even if α is not an nth root of anything in F (try to think of an example).

(c) Find the minimal polynomial of $\zeta_7 + \overline{\zeta}_7$. **Hint.** $\zeta_7^4 = \overline{\zeta}_7^3$.

(d) Write down a polynomial of degree 3 over $\mathbb{Q}$ (that is solvable by radicals but) whose splitting field is not a radical extension of $\mathbb{Q}$.