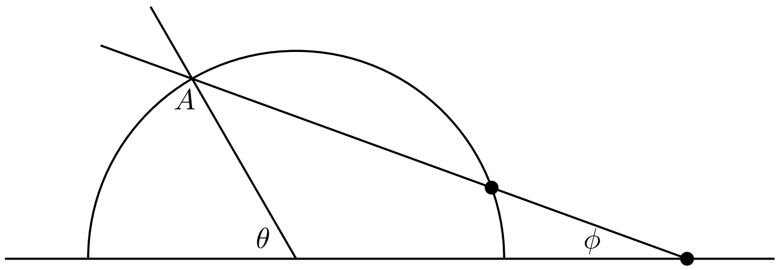
Dartmouth College Department of Mathematics
Math 81/111 Abstract Algebra
Winter 2026

Problem Set # 3 (due via Canvas upload by midnight Friday 30 January)

Reading: DF 13.3, FT pp. 22–24.

Problems:

1. Archimedes discovered a construction that trisects any given angle using a compass and "marked" straightedge. This straightedge has two special markings a distance 1 apart. The straightedge can be placed so that it passes through any already drawn point and such that both markings intersect other already drawn lines or circles.



In the picture, start with a horizontal line and another line meeting it with angle θ. Draw a circle of radius 1 around the intersection point of these two lines. The other line meets the circle at point A. Then place the marked straightedge so that it passes through the point A and such that the markings (the dots in the picture) intersect the circle and the horizontal line. Prove that the angle ϕ that the straightedge makes with the horizontal line is equal to $\theta/3$.

2. Prove that any angle θ, such that $\tan(\theta)$ is rational, can be constructed with compass and straightedge. Prove that such angles are dense in the interval $(-\pi/2, \pi/2)$.

3. Prove that an angle θ can can be trisected using compass and straightedge if and only if the polynomial $4x^3 - 3x - \cos(\theta)$ is reducible over $\mathbb{Q}(\cos(\theta))$. **Bonus.** Choose your favorite trisectable angle θ, and describe a step-by-step construction that performs the trisection; saying "I start with $\theta/3$ then triple it to make θ then I already have $\theta/3$ so I'm done" is cheating.

4. To 5-sect an angle means to divide it by 5.

(a) Prove that the angles 2π, π, $2\pi/3$, and $\pi/2$ can be 5-sected by compass and straightedge.

(b) Prove that a general angle cannot be 5-sected by compass and straightedge.

5. Let p be an odd prime.

(a) Prove that both $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_{2p})$ have degree $p-1$ over $\mathbb{Q}$.

(b) Prove that both $\mathbb{Q}(\cos(2\pi/p))$ and $\mathbb{Q}(\cos(\pi/p))$ have degree $(p-1)/2$ over $\mathbb{Q}$. **Hint.** Using the fact that $\mathbb{Q}(\cos(2\pi/p)) \subset \mathbb{R}$, prove that the minimal polynomial of ζ_p over $\mathbb{Q}(\cos(2\pi/p))$ is $x^2 - 2\cos(2\pi/p)x + 1$. Then use the tower law and the fact (which we proved in class) that $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$.

(c) Find the minimal polynomial of $\cos(2\pi/9)$ over $\mathbb{Q}$ and prove that it splits completely over $\mathbb{Q}(\cos(2\pi/9))$. **Hint.** Writing $\zeta_9 = e^{2\pi i/9}$, notice that $\zeta_9^3 = \zeta_3$ and then take real parts. Which other complex number z satisfy $z^3 = \zeta_3$?

6. For $3 \leq n < 17$, determine if a regular n-gon can be constructed by compass and straightedge. **Hint.** As we saw in class, this is equivalent to constructing ζ_n.