

DARTMOUTH COLLEGE DEPARTMENT OF MATHEMATICS  
**Math 81/111 Abstract Algebra**  
Winter 2026

Problem Set # 4 (due via Canvas upload by midnight Saturday 7 February)

**Problems:**

1. Let  $F$  be a field and  $f(x) \in F[x]$  a monic polynomial of degree  $n$ . Let  $E$  be a splitting field of  $f$  over  $F$ , so that  $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$  over  $E$ .

(a) Prove that  $\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$  is in  $F$ . This is called the **discriminant**  $\Delta(f)$  of  $f$ .

**Hint.** You might need the Fundamental Theorem of Symmetric Polynomials.

(b) Prove that  $\Delta(f) = 0$  if and only if  $f(x)$  has a repeated root in  $E$ .

(c) Prove that if  $\Delta$  is not a square in  $F$  then  $[E : F]$  is even. **Hint.** The tower law.

2. Let  $F$  be a field and let  $g(x) = x^2 + ax + b \in F[x]$ . Let  $K = F(\alpha)$ , where  $\alpha$  is a root of  $g(x)$ , so that  $g(x) = (x - \alpha)(x - \beta)$  over  $K$ . This problem is mostly review of what you already know about quadratic polynomials, so you don't need to write much!

(a) Prove that  $\Delta(g) = (\alpha - \beta)^2 = a^2 - 4b \in F$ . **Hint.** Use elementary symmetric polynomials.

(b) Assume that the characteristic of  $F$  is not 2. Prove that  $K = F(\sqrt{\Delta(g)})$ . Deduce that  $g(x)$  is irreducible over  $F$  if and only if  $\Delta(g)$  is not a square in  $F$ . Also, prove that  $g(x)$  is a square in  $F[x]$  if and only if  $\Delta(g) = 0$ . **Hint.** Use the quadratic formula.

(c) Now let  $F = \mathbb{F}_2(t)$  be the rational function field over  $\mathbb{F}_2$ . Let  $g(x) = x^2 - t \in F[x]$ . Prove that  $g(x)$  is irreducible over  $F$ , though it satisfies  $\Delta(g) = 0$ . Recall, from lecture, that  $K \cong F(\sqrt{t})$  is an example of an inseparable extension (though in lecture we only indicated how to prove the irreducibility of  $g(x)$ ). **Hint.** Proving irreducibility can either go like proving  $\sqrt{2}$  is irrational, or using Eisenstein for the ring  $\mathbb{F}_2[t]$ .

Weird stuff can happen with quadratic polynomials in characteristic 2!

3. Let  $F$  be a field and let  $f(x) = x^3 + px + q \in F[x]$ . Let  $L$  be the splitting field of  $f$ , so that  $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$  over  $L$ , for elements  $\alpha_1, \alpha_2, \alpha_3 \in L$ .

(a) Prove that  $\Delta(f) = \prod_{1 \leq i < j \leq 3} (\alpha_i - \alpha_j)^2 = -4p^3 - 27q^2 \in F$ .

(b) Let  $\alpha \in L$  be one of the roots of  $f(x)$ . Factor  $f(x) = (x - \alpha)g(x)$  over  $F(\alpha)$ , where  $g(x) \in F(\alpha)[x]$  is quadratic. Prove that  $\Delta(f) = g(\alpha)^2 \Delta(g)$ .

(c) Assume that the characteristic of  $F$  is not 2. If  $f(x)$  is irreducible and  $\alpha$  is a root of  $f(x)$ , prove that  $L = F(\alpha, \sqrt{\Delta(f)})$ . Deduce that, in this case,  $\Delta(f)$  is a square in  $F$  if and only if  $L$  has degree 3 over  $F$ . If  $f(x)$  is reducible over  $F$ , prove that  $L = F(\sqrt{\Delta(f)})$ ; for this, you cannot necessarily just assume that the root  $\alpha$  is in  $F$  in the previous case..

(d) Assume that the characteristic of  $F$  is not 2 or 3. Prove that if  $\Delta(f) = 0$  then  $L = F$ , i.e., all the roots of  $f(x)$  are in  $F$ .

(e) Write down a monic irreducible cubic polynomial over  $\mathbb{F}_3(t)$  whose discriminant is 0, and factor it over its splitting field. Write down a monic cubic polynomial over  $\mathbb{F}_2(t)$  whose discriminant is 0, and whose splitting field is nontrivial. **Hint.** Think inseparably.

- (f) Now let  $F = \mathbb{F}_2(t)$  and let  $f(x) = x^3 + tx + t$ . Prove that  $f(x)$  is irreducible over  $F$ , has nonzero square discriminant, yet its splitting field  $L$  has degree 6 over  $F$ . **Hint.** You may find it useful to use Gauss's Lemma for the ring  $F[t]$ , see Dummit and Foote, §9.3.

Weird stuff can happen with cubic polynomials in characteristics 2 and 3!

4. Let  $p$  and  $q$  be distinct prime numbers. Prove that  $\mathbb{Q}(\sqrt{p} + \sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$  and find the minimal polynomial of  $\sqrt{p} + \sqrt{q}$  over  $\mathbb{Q}$ .

5. Let  $F$  be a field of characteristic  $\neq 2$ .

(a) Let  $a_1, \dots, a_n \in F$  be distinct elements such that no product  $a_{i_1} \cdots a_{i_r}$ , with distinct indices  $i_j$ , is a square in  $F$ . Prove that  $K = F(\sqrt{a_1}, \dots, \sqrt{a_n})$  has degree  $2^n$  over  $F$ .

(b) Prove that the field  $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \dots)$  obtained by adjoining the square roots of all prime numbers to  $\mathbb{Q}$ , is an infinite degree algebraic extension of  $\mathbb{Q}$ .