

Problems:

1. Let $f(x) \in \mathbb{R}[x]$ be a monic polynomial of degree $n \geq 1$ with discriminant Δ .
 - (a) Assume that $f(x)$ has no repeated roots and let r_2 be the number of pairs of complex conjugate (nonreal) roots. Prove that the sign of Δ is $(-1)^{r_2}$.
 - (b) Let $f(x) \in \mathbb{R}[x]$ be a cubic polynomial. Prove that $\Delta \geq 0$ if and only if the roots of $f(x)$ are all real.

2. Let $\gamma = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$.
 - (a) Show that $\mathbb{Q}(\gamma)/\mathbb{Q}$ is Galois with cyclic Galois group.
 - (b) Show that $\mathbb{Q}(\gamma, i) = \mathbb{Q}(\zeta_{16})$ and compute how each element of the Galois group $\text{Gal}(\mathbb{Q}(\zeta_{16})/\mathbb{Q})$ acts on γ and i .

3. In this problem, you'll see a polynomial whose roots are solvable by radicals but whose splitting field is not itself a radical extension. Let $K = \mathbb{Q}(\zeta_7)$ and $L = \mathbb{Q}(\zeta_7 + \bar{\zeta}_7)$.
 - (a) Prove that K/\mathbb{Q} is a radical extension.
 - (b) Prove that L/\mathbb{Q} is not a radical extension. **Warning.** A simple extension $F(\alpha)$ can be radical even if α is not an n th root of anything in F (try to think of an example).
 - (c) Find the minimal polynomial of $\zeta_7 + \bar{\zeta}_7$. **Hint.** $\zeta_7^4 = \bar{\zeta}_7^3$.
 - (d) Write down a polynomial of degree 3 over \mathbb{Q} (that is solvable by radicals but) whose splitting field is not a radical extension of \mathbb{Q} .

4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible quartic polynomial and $K = \mathbb{Q}(\alpha)$ for a root α of $f(x)$. Let $G \subset S_4$ be the Galois group of the splitting field of $f(x)$ over \mathbb{Q} .
 - (a) Prove that K/\mathbb{Q} has no nontrivial intermediate subfields if and only if $G = A_4$ or $G = S_4$.
 - (b) Prove that α is constructible if and only if the normal closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$ has Galois group C_4 , V_4 , or D_8 .
 - (c) Find a nonconstructible $\alpha \in \mathbb{C}$ with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

5. Let p be a prime number and S_p the symmetric group on p things.
 - (a) Prove that an element of S_p has order p if and only if it is a p -cycle.
 - (b) Prove that S_p is generated by any choice of element of order p and a transposition. Find a composite n and a choice of an n -cycle and a transposition that do not generate S_n . **Hint.** For any n , DF 3.5 Exercise #4 says that S_n is generated by (12) and $(12 \cdots n)$. What is special about p being prime is that every element of order p in S_p is a p -cycle and every power of a p -cycle is a p -cycle (or the identity), which are facts you should verify. Hence up to conjugation and taking powers, you can assume that your transposition is (12) and that your p -cycle is $(12 \cdots)$. What then?
 - (c) Let $F \subset \mathbb{R}$ be a subfield. Prove that if $f(x) \in F[x]$ is an irreducible polynomial of degree p having $p - 2$ real roots, then the Galois group of $f(x)$ over F is isomorphic to S_p .
 - (d) Let $F \subset \mathbb{R}$ be a subfield. Prove that if $f(x) \in F[x]$ is an irreducible cubic polynomial with $\Delta < 0$, then the Galois group of $f(x)$ over F is isomorphic to S_3 .
 - (e) Prove that the Galois group of the polynomial $x^3 - x - 1$ over \mathbb{Q} is isomorphic to S_3 .
 - (f) Prove that the Galois group of the polynomial $x^5 - x^4 - x^2 - x + 1$ over \mathbb{Q} is isomorphic to S_5 . **Hint.** You are allowed to use real analysis (e.g., the intermediate value theorem), but as a challenge, try to find a purely algebraic (possibly computer-aided) way.