# RATIONAL CUBIC FOURFOLDS CONTAINING A PLANE WITH NONTRIVIAL CLIFFORD INVARIANT

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Abstract. We isolate a general class of smooth rational cubic fourfolds $X$ containing a plane whose associated quadric surface bundle does not have a rational section. Equivalently, the Brauer class of the even Clifford algebra over the discriminant cover—a K3 surface $S$ of degree 2—associated to the quadric bundle, is nontrivial. Such cubic fourfolds provide the first nontrivial corroboration of Kuznetsov's derived categorical conjecture on the rationality of cubic fourfolds containing a plane.

## Introduction

Let $X$ be a *cubic fourfold* over a field $k$, i.e., a smooth cubic hypersurface $X \subset \mathbb{P}^5_k$. Determining the (geometric) $k$-rationality of $X$ is a classical question in algebraic geometry. Some classes of geometrically rational cubic fourfolds have been described by Fano [7], Tregub [27], and Beauville–Donagi [3]. In particular, *pfaffian cubic fourfolds*, defined by pfaffians of skew-symmetric $6 \times 6$ matrices of linear forms, are rational. When $k = \mathbb{C}$, Hassett [9] describes, via lattice and Hodge theory, divisors $\mathcal{C}_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds. In particular, $\mathcal{C}_{14}$ is the closure of the locus of pfaffian cubic fourfolds and $\mathcal{C}_8$ is the locus of cubic fourfolds containing a plane. Hassett [10] identifies countably many divisors of $\mathcal{C}_8$ consisting of rational cubic fourfolds. Nevertheless, it is expected that the general cubic fourfold (and the general cubic fourfold containing a plane) is nonrational. At present, however, not a single cubic fourfold is provably nonrational.

In this work, we study a class of rational cubic fourfolds in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ not contained on the divisors of $\mathcal{C}_8$ described by Hassett. These cubic fourfolds provide the first nontrivial corroboration of Kuznetsov's derived categorical conjecture on the rationality of cubic fourfolds containing a plane.

Kuznetsov [20] establishes a semiorthogonal decomposition of the bounded derived category

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2) \rangle.$$

The category $\mathbf{A}_X$ has the remarkable property of being a 2-Calabi–Yau category, essentially a noncommutative deformation of the derived category of a K3 surface. Based on evidence from known cases as well as general categorical considerations, Kuznetsov conjectures that the category $\mathbf{A}_X$ contains all the information about the rationality of $X$.

**Conjecture** (Kuznetsov)**.** *A complex cubic fourfold $X$ is rational if and only if there exists a K3 surface $S$ and an equivalence $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S)$.*

If $X$ contains a plane, a further geometric description of $\mathbf{A}_X$ is available. Indeed, $X$ is birational to the total space of a quadric surface bundle $\pi : \widetilde{X} \to \mathbb{P}^2$ by projecting from the plane. The degeneration divisor of $\pi$ is a sextic curve $D \subset \mathbb{P}^2$ with discriminant cover $f : S \to \mathbb{P}^2$ branched along $D$. Let $\mathscr{C}_0$ be the even Clifford algebra associated to $\pi$, cf. [19] or [1, §2]. We assume that $\pi : \widetilde{X} \to \mathbb{P}^2$ has simple degeneration, hence $S$ is a smooth K3 surface of degree 2 and $\mathscr{C}_0$ defines an Azumaya algebra over $S$ by [19, Prop. 3.13]. We call the Brauer class $\beta \in \mathrm{Br}(S)$ of $\mathscr{C}_0$ the *Clifford invariant* of $X$. Via mutations, Kuznetsov [20, Thm. 4.3] establishes an equivalence $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S, \beta)$ with the bounded derived category of $\beta$-twisted sheaves on $S$.

By classical results in the theory of quadratic forms (see [1, Thm. 2.24]), $\beta$ is trivial if and only if $\pi : \widetilde{X} \to \mathbb{P}^2$ has a rational section, equivalently $\widetilde{X}$ (hence $X$) is $k(\mathbb{P}^2)$-rational. In particular, if

*Date*: June 28, 2012.

$\beta \in \mathrm{Br}(S)$ is trivial then $X$ is $k$-rational and Kuznetsov's conjecture is verified. This should be understood as the *trivial case* of Kuznetsov's conjecture for cubic fourfolds containing a plane.

**Conjecture** (Kuznetsov "containing a plane")**.** *Let $X$ be a smooth complex cubic fourfold containing a plane, $S$ the associated K3 surface of degree 2, and $\beta \in \mathrm{Br}(S)$ the Clifford invariant. Then $X$ is rational if and only if there exists a K3 surface $S'$ and an equivalence $\mathrm{D}^{\mathrm{b}}(S, \beta) \cong \mathrm{D}^{\mathrm{b}}(S')$.*

To date, this variant of Kuznetsov's conjecture is only known to hold in the trivial case (where $\beta$ is trivial and $S = S'$). Answering a question of E. Macrì and P. Stellari, we showcase a class of smooth rational cubic fourfolds containing a plane that verify this variant of Kuznetsov's conjecture in a nontrivial way, i.e., where $\beta$ is not trivial and there exists a different K3 surface $S'$ and an equivalence $\mathrm{D}^{\mathrm{b}}(S, \beta) \cong \mathrm{D}^{\mathrm{b}}(S')$. The existence of such fourfolds is not *a priori* clear: while a very general cubic fourfold containing a plane has nontrivial Clifford invariant, the existence of *rational* such fourfolds is only intimated in the literature.

First, we give explicit conditions for $X \in \mathcal{C}_8$ to have nontrivial Clifford invariant $\beta$, as follows.

**Theorem 1.** *Let $X$ be a smooth cubic fourfold containing a plane, $S$ the associated K3 surface of degree 2, and $\beta \in \mathrm{Br}(S)$ the Clifford invariant. If $S$ has geometric Picard rank 2 and even Néron–Severi discriminant then the Clifford invariant $\beta \in \mathrm{Br}(S)$ of $X$ is nontrivial.*

Next, we construct a *pfaffian* cubic fourfold with nontrivial Clifford invariant. Such a fourfold is rational and *nontrivially* satisfies Kuznetsov's conjecture on cubic fourfolds containing a plane.

**Theorem 2.** *There exist smooth pfaffian cubic fourfolds $X$ containing a plane with nontrivial Clifford invariant $\beta \in \mathrm{Br}(S)$. Any such $X$ is rational and there exists a K3 surface $S'$ of degree 14 and a nontrivial twisted derived equivalence $\mathrm{D}^{\mathrm{b}}(S, \beta) \cong \mathrm{D}^{\mathrm{b}}(S')$.*

The nontriviality of the Brauer class in Theorem 1 is proved via Hodge theory (see Proposition 6). The existence of pfaffian cubic fourfolds in Theorem 2 is proved by outlining geometric conditions on the associated K3 surface $S$ that imply the nontriviality of $\beta$ (see Proposition 7), then assembling a pfaffian cubic fourfold (see Theorem 14) explicitly verifying these conditions. For the verification, we were aided by `Magma` [5], adapting some of the computational techniques developed in [12].

Finally, putting our results in context, we prove that the cubic fourfolds considered in Theorem 2 contain a quartic scroll labeling (see Proposition 9) and have groups of algebraic 2-cycles $A(X)$ of minimal rank. We obtain the following complete numerical classification for such cubic fourfolds.

**Theorem 3.** *Let $X$ be a smooth complex cubic fourfold containing a plane and a quartic scroll labeling with $A(X)$ of minimal rank. The intersection lattice of $X$ is determined by its discriminant $d_X$, which can take on each of the possible values $\{21, 29, 32, 36, 37\}$. The Clifford invariant of $X$ is trivial if and only if $d_X$ is odd.*

This answers a question of F. Charles on cubic fourfolds in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ with trivial Clifford invariant (see Propositions 10 and 11). Our classification result employs recent work of Mayanskiy [23].

Throughout, we are guided by a remark of Hassett [10, Rem. 4.3], suggesting that rational cubic fourfolds containing a plane with nontrivial Clifford invariant ought to lie in $\mathcal{C}_8 \cap \mathcal{C}_{14}$. While the locus of pfaffian cubic fourfolds is dense in $\mathcal{C}_{14}$, it is not clear that the locus of pfaffians containing a plane is dense in $\mathcal{C}_8 \cap \mathcal{C}_{14}$. The cubic fourfolds in Theorems 2 and 3 affirm Hassett's suggestion.

**Acknowledgments.** Much of this work has been developed during visits of the authors at the Max Planck Institut für Mathematik in Bonn, Universität Duisburg–Essen, Université Rennes 1, ETH Zürich, and Rice University. The hospitality of each institute is warmly acknowledged. The first and fourth authors are partially supported by NSF grant MSPRF DMS-0903039 and DMS-1103659, respectively. The second author was partially supported by the SFB/TR 45 'Periods, moduli spaces, and arithmetic of algebraic varieties'. The authors would specifically like to thank F. Charles, J.-L. Colliot-Thélène, B. Hassett, M.-A. Knus, E. Macrì, R. Parimala, and V. Suresh for many helpful discussions.

## 1. Nontriviality criteria for Clifford invariants

In this section, by means of straightforward lattice-theoretic calculations, we describe a class of cubic fourfolds containing a plane with nontrivial Clifford invariant.

If $(H, b)$ is a $\mathbb{Z}$-lattice and $A \subset H$, then the orthogonal complement $A^\perp = \{v \in H : b(v, A) = 0\}$ is a *saturated* sublattice (i.e., $A^\perp = A^\perp \otimes_\mathbb{Z} \mathbb{Q} \cap H$) and is thus a *primitive* sublattice (i.e., $H/A^\perp$ is torsion free). Denote by $d(H, b) \in \mathbb{Z}$ the *discriminant*, i.e., the determinant of the Gram matrix.

Let $X$ be a smooth cubic fourfold over $\mathbb{C}$. The integral Hodge conjecture holds for $X$ (by [24], [32], cf. [31, Thm. 18]) and we denote by $A(X) = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ the lattice of integral middle Hodge classes, which are all algebraic.

Now suppose that $X$ contains a plane $P$ and let $\pi : \widetilde{X} \to \mathbb{P}^2$ be the quadric surface bundle defined by blowing up and projecting away from $P$. Let $\mathscr{C}_0$ be the even Clifford algebra associated to $\pi$, cf. [19] or [1, §2]. Throughout, we always assume that $\pi$ has *simple degeneration*, i.e., the fibers of $\pi$ have at most isolated singularities. This is equivalent to the condition that $X$ doesn't contain another plane intersecting $P$; see [29, Lemme 2]. This implies that the degeneration divisor $D \subset \mathbb{P}^2$ is a smooth sextic curve, the *discriminant cover* $f : S \to \mathbb{P}^2$ branched along $D$ is a smooth K3 surface of degree 2, and that $\mathscr{C}_0$ defines an Azumaya quaternion algebra over $S$, cf. [19, Prop. 3.13]. We refer to the Brauer class $\beta \in \mathrm{Br}(S)[2]$ of $\mathscr{C}_0$ as the *Clifford invariant* of $X$.

Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class associated to the embedding $X \subset \mathbb{P}^5$. The *transcendental* lattice $T(X)$, the *nonspecial cohomology* lattice $K$, and the *primitive cohomology* lattice $H^4(X, \mathbb{Z})_0$ are the orthogonal complements (with respect to the cup product polarization $b_X$) of $A(X)$, $\langle h^2, P\rangle$, and $\langle h^2\rangle$ inside $H^4(X, \mathbb{Z})$, respectively. Thus $T(X) \subset K \subset H^4(X, \mathbb{Z})_0$. We have that $T(X) = K$ for a very general cubic fourfold, cf. the proof of [29, Prop. 2]. There are natural polarized Hodge structures on $T(X)$, $K$, and $H^4(X, \mathbb{Z})_0$ given by restriction from $H^4(X, \mathbb{Z})$.

Similarly, let $S$ be a smooth integral projective surface over $\mathbb{C}$ and $\mathrm{NS}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$ its Néron–Severi lattice. Let $h_1 \in \mathrm{NS}(S)$ be a fixed anisotropic class. The *transcendental* lattice $T(S)$ and the *primitive cohomology* $H^2(S, \mathbb{Z})_0$ are the orthogonal complements (with respect to the cup product polarization $b_S$) of $\mathrm{NS}(S)$ and $\langle h_1\rangle$ inside $H^2(S, \mathbb{Z})$, respectively. If $f : S \to \mathbb{P}^2$ is a double cover, then we take $h_1$ to be the class of $f^*\mathscr{O}_{\mathbb{P}^2}(1)$.

Let $F(X)$ be the Fano variety of lines in $X$ and $W \subset F(X)$ the divisor consisting of lines meeting $P$. Then $W$ is identified with the relative Hilbert scheme of lines in the fibers of $\pi$. Its Stein factorization $W \xrightarrow{p} S \xrightarrow{f} \mathbb{P}^2$ displays $W$ as a smooth conic bundle over the discriminant cover. Then the Abel–Jacobi map

$$\Phi : H^4(X, \mathbb{Z}) \to H^2(W, \mathbb{Z})$$

becomes an isomorphism of $\mathbb{Q}$-Hodge structures $\Phi : H^4(X, \mathbb{Q}) \to H^2(W, \mathbb{Q})(-1)$; see [29, Prop. 1]. Finally, $p : W \to S$ is a smooth conic bundle and there is an injective (see [30, Lemma 7.28]) morphism of Hodge structures $p^* : H^2(S, \mathbb{Z}) \to H^2(W, \mathbb{Z})$. We recall a result of Voisin [29, Prop. 2].

**Proposition 4.** *Let $X$ be a smooth cubic fourfold containing a plane. Then $\Phi(K) \subset p^*H^2(S, \mathbb{Z})_0(-1)$ is a polarized Hodge substructure of index 2.*

*Proof.* That $\Phi(K) \subset p^*H^2(S, \mathbb{Z})_0$ is an inclusion of index 2 is proved in [29, Prop. 2]. We now verify that the inclusion respects the Hodge filtrations. The Hodge filtration of $\Phi(K) \otimes_\mathbb{Z} \mathbb{C}$ is that induced from $H^2(W, \mathbb{C})(-1)$ since $\Phi$ is an isomorphism of $\mathbb{Q}$-Hodge structures. On the other hand, since $p : W \to S$ is a smooth conic bundle, $R^1p_*\mathbb{C} = 0$. Hence $p^* : H^2(S, \mathbb{C}) \to H^2(W, \mathbb{C})$ is injective by the Leray spectral sequence and $p^*H^{p,2-p}(S) = p^*H^2(S, \mathbb{C}) \cap H^{p,2-p}(W)$. Thus the Hodge filtration of $p^*H^2(S, \mathbb{C})(-1)$ is induced from $H^2(W, \mathbb{C})(-1)$, and similarly for primitive cohomology. In particular, the inclusion $\Phi(K) \subset p^*H^2(S, \mathbb{Z})_0(-1)$ is a morphism of Hodge structures. Finally, by [29, Prop. 2], we have that $b_X(x, y) = -b_S(\Phi(x), \Phi(y))$ for $x, y \in K$, and thus the inclusion also preserves the polarizations. □

By abuse of notation (of which we are already guilty), for $x \in K$, we will consider $\Phi(x)$ as an element of $p^*H^2(S, \mathbb{Z})_0(-1)$ without explicitly mentioning so.

**Corollary 5.** *Let $X$ be a smooth cubic fourfold containing a plane. Then $\Phi(T(X)) \subset p^*T(S)(-1)$ is a sublattice of index $\epsilon$ dividing 2. In particular,* $\operatorname{rk} A(X) = \operatorname{rk} \operatorname{NS}(S) + 1$ *and* $d(A(X)) = 2^{2(\epsilon-1)} d(\operatorname{NS}(S))$.

*Proof.* By the saturation property, $T(X)$ and $T(S)$ coincide with the orthogonal complement of $A(X) \cap K$ in $K$ and $\operatorname{NS}(S) \cap H^2(S,\mathbb{Z})_0$ in $H^2(S,\mathbb{Z})_0$, respectively. Now, for $x \in T(X)$ and $a \in \operatorname{NS}(S)_0$, we have

$$b_S(\Phi(x), a) = -\frac{1}{2}\Phi(x).g.p^*a = -\frac{1}{2} b_X(x, {}^t\Phi(g.p^*a)) = 0$$

by [29, Lemme 3] and the fact that ${}^t\Phi(g.p^*a) \in A(X)$ (here, $g \in H^2(W,\mathbb{Z})$ is the pullback of the hyperplane class from the canonical grassmannian embedding), which follows since ${}^t\Phi : H^4(W,\mathbb{Z}) \cong H_2(W,\mathbb{Z}) \to H_4(X,\mathbb{Z}) \cong H^4(X,\mathbb{Z})$ preserves the Hodge structure by the same argument as in the proof of Proposition 4. Therefore $\Phi(T(X)) \subset p^*T(S)(-1)$.

Since $T(X) \subset K$ and $T(S)(-1) \subset H^2(S,\mathbb{Z})_0(-1)$ are saturated (hence primitive) sublattices, an application of the snake lemma shows that $p^*T(S)(-1)/\Phi(T(X)) \subset p^*H^2(S,\mathbb{Z})_0/\Phi(K) \cong \mathbb{Z}/2\mathbb{Z}$, hence the index of $\Phi(T(X))$ in $p^*T(S)(-1)$ divides 2.

We now verify the final claims. We have $\operatorname{rk} K = \operatorname{rk} H^2(X,\mathbb{Z}) - 2 = \operatorname{rk} T(X) + \operatorname{rk} A(X) - 2$ and $\operatorname{rk} H^2(S,\mathbb{Z})_0 = \operatorname{rk} H^2(S,\mathbb{Z}) - 1 = \operatorname{rk} T(S) + \operatorname{rk} \operatorname{NS}(S) - 1$ (since $P$, $h^2$, and $h_1$ are anisotropic vectors, respectively), while $\operatorname{rk} K = \operatorname{rk} H^2(S,\mathbb{Z})_0$ and $\operatorname{rk} T(X) = \operatorname{rk} T(S)$ by Proposition 4 and the above, respectively. The claim concerning the discriminant then follows by standard lattice theory. □

Let $Q \in A(X)$ be the class of a fiber of the quadric surface bundle $\pi : \widetilde{X} \to \mathbb{P}^2$. Then $P + Q = h^2$, see [29, §1].

**Proposition 6.** *Let $X$ be a smooth cubic fourfold containing a plane $P$. If $A(X)$ has rank 3 and even discriminant then the Clifford invariant $\beta \in \operatorname{Br}(S)$ of $X$ is nontrivial.*

*Proof.* The Clifford invariant $\beta \in \operatorname{Br}(S)$ associated to the quadric surface bundle $\pi : \widetilde{X} \to \mathbb{P}^2$ is trivial if and only if $\pi$ has a rational section; see [15, Thm. 6.3] or [26, 2 Thm. 14.1, Lemma 14.2]. Such a section exists if and only if there exists an algebraic cycle $R \in A(X)$ such that $R.Q = 1$; see [10, Thm. 3.1] or [20, Prop. 4.7].

Suppose that such a cycle $R$ exists and consider the sublattice $\langle h^2, Q, R\rangle \subset A(X)$. Its intersection form has Gram matrix

$$\begin{array}{c|ccc} & h^2 & Q & R \\ h^2 & 3 & 2 & x \\ Q & 2 & 4 & 1 \\ R & x & 1 & y \end{array} \tag{1}$$

for some $x, y \in \mathbb{Z}$. The determinant of this matrix is always congruent to 5 modulo 8, so this lattice cannot be a finite index sublattice of $A(X)$, which has even discriminant by hypothesis. Hence no such 2-cycle $R$ exists and thus $\beta$ is nontrivial. The final claim follows directly from Corollary 5. □

*Proof of Theorem 1.* If the associated K3 surface $S$ of degree 2 has Picard rank 2 and even Néron–Severi discriminant, then $A(X)$ has rank 3 and even discriminant by Corollary 5, and we are done by Proposition 6. □

We now provide an explicit geometric condition for the nontriviality of the Clifford invariant, which will be necessary in §4. We say that a cubic fourfold $X$ containing a plane has a *tangent conic* if there exists a conic $C \subset \mathbb{P}^2$ tangent to the discriminant curve $D \subset \mathbb{P}^2$ of the associated quadric surface bundle.

**Proposition 7.** *Let $X$ be a smooth cubic fourfold containing a plane. Let $S$ be the associated K3 surface of degree 2 and $\beta \in \operatorname{Br}(S)$ the Clifford invariant. If $X$ has a tangent conic and $S$ has Picard rank 2 then $\beta$ is nontrivial.*

*Proof.* Consider the pull back of the cycle class of $C$ to $S$ via the discriminant double cover $f : S \to \mathbb{P}^2$. Then $f^*C$ has two components $C_1$ and $C_2$. The sublattice of the Néron–Severi lattice of $S$ generated by $h_1 = f^*\mathscr{O}_{\mathbb{P}^2}(1) = (C_1 + C_2)/2$ and $C_1$ has intersection form with Gram matrix

| | $h_1$ | $C_1$ |
|---|---|---|
| $h_1$ | 2 | 2 |
| $C_1$ | 2 | $-2$ |

having determinant $-8$. As $S$ has Picard rank 2, then the entire Néron–Severi lattice is in fact generated by $h_1$ and $C_1$ (see [6, §2] for further details) and we can apply Proposition 6 to conclude the nontriviality of the Clifford invariant. ☐

*Remark* 8. Kuznetsov's conjecture implies that the very general cubic fourfold containing a plane is not rational. In particular, any rational cubic fourfold containing a plane should have associated K3 surface of degree 2 with Picard rank at least 2. A stronger statement is true, namely there exists no K3 surface $S'$ with $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S')$; see [20, Prop. 4.8].

## 2. The Clifford invariant on $\mathcal{C}_8 \cap \mathcal{C}_{14}$

In this section, we first prove that cubic fourfolds $X$ containing a tangent conic with $A(X)$ of rank 3 (i.e., those considered in Proposition 7) are contained in $\mathcal{C}_8 \cap \mathcal{C}_{14}$. Recall that such fourfolds have a nontrivial Clifford invariant. One may wonder if the Clifford invariant is nontrivial for *every* cubic fourfold $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ with $A(X)$ of rank 3. However, we find three classes of such cubic fourfolds with *trivial* Clifford invariant (see Proposition 11), answering a question of F. Charles.

We say that a cubic fourfold $X$ has a *quartic scroll labeling* if $A(X)$ contains a primitive sublattice $\langle h^2, T\rangle$ with Gram matrix

(2)

| | $h^2$ | $T$ |
|---|---|---|
| $h^2$ | 3 | 4 |
| $T$ | 4 | 10 |

The name is suggestive: if $X$ contains a rational normal quartic scroll then $X$ contains a quartic scroll labeling; see [9, §4.1.3]. If $X$ has a quartic scroll labeling then $X \in \mathcal{C}_{14}$.

**Proposition 9.** *Let $X$ be a smooth cubic fourfold containing a plane $P$ and a tangent conic such that $A(X)$ has rank 3. Then $X$ has a quartic scroll labeling such that $\langle h^2, P, T\rangle$ has Gram matrix*

| | $h^2$ | $P$ | $T$ |
|---|---|---|---|
| $h^2$ | 3 | 1 | 4 |
| $P$ | 1 | 3 | 0 |
| $T$ | 4 | 0 | 10 |

*Proof.* Since $X$ contains a tangent conic, $A(X)$ has discriminant 8 or 32 and $X$ has nontrivial Clifford invariant by Corollary 5 and Proposition 7. As the sublattice $\langle h^2, P\rangle \subset A(X)$ is primitive, we can choose a class $T \in A(X)$ such that $\langle h^2, P, T\rangle \subset A(X)$ has discriminant 32. Adjusting $T$ by a multiple of $P$, we can assume that $h^2.T = 4$. Write $\tau = P.T$. If $\tau$ is odd, then $(P + T).Q = -\tau$ is odd, hence the Clifford invariant is trivial by an application of the criteria in [10, Thm. 3.1] or [20, Prop. 4.7] (cf. the proof of Proposition 6). This is a contradiction, hence $\tau$ is even.

Adjusting $T$ by multiples of $h^2 - 3P$ keeps $h^2.T = 4$ and adjusts $\tau$ by multiples of 8. The discriminant being 32, we are left with two possible choices for the Gram matrix of $\langle h^2, P, T\rangle$ up to isomorphism:

| | $h^2$ | $P$ | $T$ |
|---|---|---|---|
| $h^2$ | 3 | 1 | 4 |
| $P$ | 1 | 3 | 0 |
| $T$ | 4 | 0 | 10 |

| | $h^2$ | $P$ | $T$ |
|---|---|---|---|
| $h^2$ | 3 | 1 | 4 |
| $P$ | 1 | 3 | 4 |
| $T$ | 4 | 4 | 12 |

In these cases, we compute that $K \cap \langle h^2, P, T\rangle$, i.e., the orthogonal complement of $\langle h^2, P\rangle$ in $\langle h^2, P, T\rangle$, is generated by $3h^2 - P - 2T$ and $h^2 + P - T$, and has discriminant $-4$ and 4, respectively.

Let $S$ be the associated K3 surface of degree 2. We calculate that $\mathrm{NS}(S) \cap H^2(S,\mathbb{Z})_0$, i.e., the orthogonal complement of $\langle h_1 \rangle$ in $\mathrm{NS}(S)$, is generated by $h_1 - C_1$ and has discriminant $-4$. Arguing as in the proof of Corollary 5, there's an lattice inclusion $\Phi(K \cap \langle h^2, P, T\rangle) \subset \mathrm{NS}(S) \cap H^2(S,\mathbb{Z})_0(-1)$ having index dividing 2, which rules out the second case above by comparing discriminants. ☐

We will consider, as suggested by Proposition 9, cubic fourfolds $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ containing a plane $P$ and a quartic scroll marking defined by $T$ with $A(X)$ having rank 3. The intersection form of the sublattice $\langle h^2, P, T\rangle \subset A(X)$ then has Gram matrix

$$\begin{array}{cccc} & h^2 & P & T \\ h^2 & 3 & 1 & 4 \\ P & 1 & 3 & \tau \\ T & 4 & \tau & 10 \end{array} \tag{3}$$

for some $\tau \in \mathbb{Z}$ depending on $X$. There may be *a priori* restrictions on the possible values of $\tau$. By Proposition 9, $\tau = 0$ if $X$ has a tangent conic. We will use recent work of Mayanskiy [23, Thm. 6.1, Rem. 6.3] to classify exactly which values of $\tau$ are supported by cubic fourfolds.

**Proposition 10.** *If $X$ is a smooth cubic fourfold containing a plane and a quartic scroll labeling such that $A(X)$ has rank 3, then $\tau$ can take on each of the possible values $\{-1, 0, 1, 2, 3\}$.*

*Proof.* Let $A = h^2\mathbb{Z} \oplus P\mathbb{Z} \oplus T\mathbb{Z}$ and let $(A, b)$ be the lattice whose bilinear form has Gram matrix (3). If $X$ is a smooth cubic fourfold, then $A(X)$ is positive definite by the Riemann bilinear relations. Thus in order to be realized as a sublattice of some $A(X)$, the lattice $(A, b)$ must be positive definite, which by Sylvester's criterion, is equivalent to $(A, b)$ having positive discriminant. As $d(A, b) = -3\tau^2 + 8\tau + 32$, the only values of $\tau$ making a positive discriminant are $\{-2, -1, 0, 1, 2, 3, 4\}$ with corresponding discriminants $\{4, 21, 32, 37, 36, 29, 16\}$.

We now attempt to verify the conditions (1)–(6) of [23, Thm. 6.1] in each of these cases. Condition (1) is true by definition. For condition (2), the vectors $(1, -3, 0)$ and $(0, -4, 1)$ form a basis for $A_0 = \langle h^2 \rangle^\perp$. Then letting $v = (x, -3x - 4y, y) \in A_0$, we see that

$$b(v, v) = 2\big(12x^2 + (36 - 3\tau)xy + (29 - 4\tau)y^2\big) \tag{4}$$

is even. For condition (5), letting $w = (x, y, z) \in A$, we compute that

$$b(h^2, w)^2 - b(w, w) = 2\big(3x^2 - y^2 + z^2 + 2xy + 8xz + (4 - \tau)yz\big) \tag{5}$$

is even. For conditions (3)–(4), given each $\tau \in \{-1, 0, 1, 2, 3\}$, we use standard Diophantine techniques to prove the nonexistence of *short* and *long roots* (see [21, §2] for definitions) of (4). For $\tau = -2$, we find short roots $(-2, 2, 1)$ and $(2, -10, 1)$; for $\tau = 4$, we find short roots $\pm(1, 1, -1)$.

Finally, for condition (6), let $q_K : A^*/A \to \mathbb{Q}/2\mathbb{Z}$ be the discriminant form of (5), restricted to the discriminant group $A^*/A$ of the lattice $A$. Appealing to Nikulin [25, Cor. 1.10.2], it suffices to check that the signature satisfies $\mathrm{sgn}(q_K) \equiv 0 \bmod 8$; cf. [23, Rem. 6.3]. Employing the notation of [25, Prop. 1.8.1], we compute the finite quadratic form $q_K$ in each remaining case:

(6)

| $\tau$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $d(A)$ | 21 | 32 | 37 | 36 | 29 |
| $A^*/A$ | $\mathbb{Z}/21\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/37\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}$ | $\mathbb{Z}/29\mathbb{Z}$ |
| $q_K$ | $q_1^3(3) \oplus q_1^7(7)$ | $q_3^2(2) \oplus q_1^2(2^4)$ | $q_\theta^{37}(37)$ | $q_3^2(2) \oplus q_1^2(2) \oplus q_1^3(3^2)$ | $q_\theta^{29}(29)$ |

where $\theta$ represents a nonsquare class modulo the respective odd prime. In each case of (6), we verify the signature condition using the formulas in [25, Prop. 1.11.2]. In total, we've proved a stronger statement, that for each $\tau \in \{-1, 0, 1, 2, 3\}$ the lattice $A$ arises as the full lattice $A = A(X)$ for a smooth cubic fourfold $X$. ☐

We now address the question of the (non)triviality of the Clifford invariant.

**Proposition 11.** *Let $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ be a smooth cubic fourfold containing a plane and a quartic scroll labeling such that $A(X)$ has rank 3. The Clifford invariant is trivial if and only if $\tau$ is odd.*

*Proof.* If $\tau$ is odd then, as in the proof of Proposition 9, $(P+T).Q = -\tau$ is odd, hence the Clifford invariant $\beta \in \mathrm{Br}(S)$ is trivial by an application of the criteria in [10, Thm. 3.1] or [20, Prop. 4.7] (cf. the proof of Proposition 6).

If $\beta$ is trivial, then there exists a class $R' \in A(X)$ such that $Q.R' = 1$. We can choose $R$ in the saturation of $\langle h^2, P, R'\rangle \subset H^4(X,\mathbb{Z})$ such that $\langle h^2, P, R\rangle \subset H^4(X,\mathbb{Z})$ is saturated and still $Q.R = 1$; cf. proof of [10, Lemma 4.4]. Thus $A(X) = \langle h^2, P, R\rangle$ since $\operatorname{rk} A(X) = 3$ and $A(X) \subset H^4(X,\mathbb{Z})$ is saturated. By Proposition 6, we must have $d(A(X)) \equiv 5 \bmod 8$. Thus the discriminant of $\langle h^2, P, R\rangle$ must be of the form $2^{2e}d$, where $d \equiv 5 \bmod 8$. Dealing with each congruence class of $\tau$ modulo 16, this is only possible when $\tau$ is odd or $\tau \equiv 12 \bmod 16$. This last case never occurs by Proposition 10. ☐

*Proof of Theorem 3.* We combine Propositions 10 and 11 with the values in table (6). ☐

By Proposition 10, we have isolated three classes of smooth cubic fourfold $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ with *trivial* Clifford invariant. In particular, such cubic fourfolds are rational and verify Kuznetsov's conjecture; see [20, Prop. 4.7]. However, we do not know whether any such $X$ is pfaffian.

## 3. The twisted derived equivalence

Homological projective duality (HPD) can be used to obtain a significant semiorthogonal decomposition of the derived category of a pfaffian cubic fourfold. As the universal pfaffian variety is singular, a noncommutative resolution of singularities is required to establish HPD in this case. A *noncommutative resolution of singularities* of a scheme $Y$ is a coherent $\mathscr{O}_Y$-algebra $\mathscr{R}$ with finite homological dimension that is generically a matrix algebra (these properties translate to "smoothness" and "birational to $Y$" from the categorical language). We refer to [17] for details on HPD.

**Theorem 12** ([16]). *Let $W$ be a $\mathbb{C}$-vector space of dimension 6 and $Y \subset \mathbb{P}(\wedge^2 W^\vee)$ the universal pfaffian cubic hypersurface. There exists a noncommutative resolution of singularities $(Y,\mathscr{R})$ that is HP dual to the grassmannian $\mathrm{Gr}(2,W)$. In particular, the bounded derived category of a smooth pfaffian cubic fourfold $X$ admits a semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(S'), \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2)\rangle,$$

*where $S'$ is a smooth K3 surface of degree 14. In particular, $\mathbf{A}_X \cong \mathrm{D^b}(S')$.*

*Proof.* The relevant noncommutative resolution of singularities $\mathscr{R}$ of $Y$ is constructed in [18]. The HP duality is established in [16, Thm. 1]. The semiorthogonal decomposition is constructed as follows. Any pfaffian cubic fourfold $X$ is an intersection of $Y \subset \mathbb{P}(\wedge^2 W^\vee) = \mathbb{P}^{14}$ with a linear subspace $\mathbb{P}^5 \subset \mathbb{P}^{14}$. If $X$ is smooth, then $\mathscr{R}|_X$ is Morita-equivalent to $\mathscr{O}_X$. Via classical projective duality, $Y \subset \mathbb{P}^{14}$ corresponds to $\mathbb{G}(2,W) \subset \check{\mathbb{P}}^{14}$ while $\mathbb{P}^5 \subset \mathbb{P}^{14}$ corresponds to a linear subspace $\mathbb{P}^8 \subset \check{\mathbb{P}}^{14}$. The intersection of $\mathbb{G}(2,W)$ and $\mathbb{P}^8$ inside $\check{\mathbb{P}}^{14}$ is a K3 surface $S'$ of degree 14. Kuznetsov [16, Thm. 2] describes a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathscr{O}_X(-3), \mathscr{O}_X(-2), \mathscr{O}_X(-1), \mathrm{D^b}(S')\rangle.$$

To obtain the desired semiorthogonal decomposition and the equivalence $\mathbf{A}_X \cong \mathrm{D^b}(S')$, we act on $\mathrm{D^b}(X)$ by the autoequivalence $-\otimes \mathscr{O}_X(3)$, then mutate the image of $\mathrm{D^b}(S')$ to the left with respect to its left orthogonal complement; see [4]. This displays the left orthogonal complement of $\langle \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2)\rangle$, which is $\mathbf{A}_X$ by definition, as a category equivalent to $\mathrm{D^b}(S')$. ☐

Finally, assuming the result in §4, we can give a proof of Theorem 2.

*Proof of Theorem 2.* Let $X$ be a smooth complex pfaffian cubic fourfold containing a plane, $S$ the associated K3 surface of degree 2, $\beta \in \mathrm{Br}(S)$ the Clifford invariant, and $S'$ the K3 surface of degree 14 arising from Theorem 12 via projective duality. Then by [20, Thm. 4.3] and Theorem 12, the category $\mathbf{A}_X$ is equivalent to both $\mathrm{D^b}(S,\beta)$ and $\mathrm{D^b}(S')$.

The cubic fourfold $X$ is rational, being pfaffian. The existence of such cubic fourfolds with $\beta$ nontrivial is guaranteed by Theorem 14. As $\beta$ is nontrivial, there is a nontrivial twisted derived equivalence $\mathrm{D^b}(S,\beta) \cong \mathrm{D^b}(S')$ between K3 surfaces of degree 2 and 14. ☐

*Remark* 13. By [13, Rem. 7.10], given any K3 surface $S$ and any nontrivial $\beta \in \mathrm{Br}(S)$, there is *no* equivalence between $\mathrm{D^b}(S,\beta)$ and $\mathrm{D^b}(S)$. Thus any $X$ as in Theorem 2 validates Kuznetsov's conjecture on the rationality of cubic fourfolds containing a plane, but not via the K3 surface $S$.

## 4. A pfaffian containing a plane

In this section, we provide a smooth pfaffian cubic fourfold containing a plane and satisfying the hypotheses of Proposition 6, hence having nontrivial Clifford invariant.

**Theorem 14.** *Let $A$ be the $6\times 6$ antisymmetric matrix*

$$\begin{pmatrix} 0 & y+u & x+y+u & u & z & y+u+v \\ & 0 & x+y+z & x+z+u+w & y+z+u+v+w & x+y+z+u+v+w \\ & & 0 & x+y+u+w & x+y+u+v+w & x+y+z+v+w \\ & & & 0 & x+u+v+w & x+u+w \\ & & & & 0 & z+u+w \\ & & & & & 0 \end{pmatrix}$$

*of linear forms in $\mathbb{Q}[x,y,z,u,v,w]$ and let $X \subset \mathbb{P}^5$ be the cubic fourfold defined by the vanishing of the pfaffian of $A$:*

$$\begin{aligned}&(x-4y-z)u^2+(-x-3y)uv+(x-3y)uw+(x-2y-z)vw-2yv^2+xw^2\\&+(2x^2+xz-4y^2+2z^2)u+(x^2-xy-3y^2+yz-z^2)v+(2x^2+xy+3xz-3y^2+yz)w\\&+x^3+x^2y+2x^2z-xy^2+xz^2-y^3+yz^2-z^3.\end{aligned}$$

*Then:*

a) *$X$ is smooth, rational, and contains the plane $P=\{x=y=z=0\}$.*
b) *The degeneration divisor $D\subset\mathbb{P}^2$ of the associated quadric surface bundle $\pi:\widetilde{X}\to\mathbb{P}^2$ is the sextic curve given by the vanishing of:*

$$\begin{aligned}d = {}&x^6+6x^5y+12x^5z+x^4y^2+22x^4yz+28x^3y^3-38x^3y^2z+46x^3yz^2+4x^3z^3\\&+24x^2y^4-4x^2y^3z-37x^2y^2z^2-36x^2yz^3-4x^2z^4+48xy^4z-24xy^3z^2\\&+34xy^2z^3+4xyz^4+20y^5z+20y^4z^2-8y^3z^3-11y^2z^4-4yz^5.\end{aligned}$$

*This curve is smooth; in particular, $\pi$ has simple degeneration and the discriminant cover is a smooth K3 surface $S$ of degree 2.*

c) *The conic $C\subset\mathbb{P}^2$ defined by the vanishing of $x^2+yz$ is tangent to the degeneration divisor $D$ at six points (five of which are distinct).*
d) *The K3 surface $S$ has (geometric) Picard rank 2.*

*In particular, the Clifford invariant of $X$ is geometrically nontrivial.*

*Proof.* Verifying smoothness of $X$ and $D$ is a straightforward application of the jacobian criterion, while the inclusion $P\subset X$ is checked by inspecting the expression for $\mathrm{pf}(A)$; every monomial is divisible by $x$, $y$ or $z$. Rationality comes from being a pfaffian cubic fourfold; see [27]. The smoothness of $D$ implies that $\pi$ has simple degeneration; see [12, Rem. 7.1] or [1, Rem. 2.6]. This establishes parts *a)* and *b)*.

For part *c)*, note that we can write the equation for the degeneration divisor as $d=(x^2+yz)f+g^2$, where

$$\begin{aligned}f &= x^4+6x^3y+12x^3z+x^2y^2+21x^2yz-25x^2z^2+28xy^3\\&\quad -24xy^2z+34xyz^2+4xz^3+20y^4-5y^3z-8y^2z^2-11yz^3-4z^4.\\ g &= 2xy^2+5y^2z-5x^2z.\end{aligned}$$

Hence the conic $C\subset\mathbb{P}^2$ defined by $x^2+yz$ is tangent to $D$ along the zero-dimensional scheme of length 6 given by the intersection of $C$ and the vanishing of $g$.

For part *d)*, the surface $S$ is the smooth sextic in $\mathbb{P}(1,1,1,3) = \operatorname{Proj}\mathbb{Q}[x,y,z,w]$ given by

$$w^2 = d(x,y,z),$$

which is the double cover $\mathbb{P}^2$ branched along the discriminant divisor $D$. In these coordinates, the discriminant cover $f : S \to \mathbb{P}^2$ is simply the restriction to $S$ of the projection $\mathbb{P}(1,1,1,3) \dashrightarrow \mathbb{P}^2$ away from the hyperplane $\{w = 0\}$. Let $C \subset \mathbb{P}^2$ be the conic from part *d)*. As discussed in Proposition 7, the curve $f^*C$ consists of two $(-2)$-curves $C_1$ and $C_2$. These curves generate a sublattice of $\mathrm{NS}(S)$ of rank 2. Hence $\rho(\overline{S}) \geq \rho(S) \geq 2$, where $\overline{S} = S \times_{\mathbb{Q}} \mathbb{C}$.

We show next that $\rho(\overline{S}) \leq 2$. Write $S_p$ for the reduction mod $p$ of $S$ and $\overline{S}_p = S_p \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Let $\ell \neq 3$ be a prime and write $\phi(t)$ for the characteristic polynomial of the action of absolute Frobenius on $H^2_{\text{ét}}(\overline{S}_3, \mathbb{Q}_\ell)$. Then $\rho(\overline{S}_3)$ is bounded above by the number of roots of $\phi(t)$ that are of the form $3\zeta$, where $\zeta$ is a root of unity [28, Prop. 2.3]. Combining the Lefschetz trace formula with Newton's identities and the functional equation that $\phi(t)$ satisfies, it is possible calculate $\phi(t)$ from knowledge of $\#S(\mathbb{F}_{3^n})$ for $1 \leq n \leq 11$; see [28] for details.

Let $\widetilde{\phi}(t) = 3^{-22}\phi(3t)$, so that the number of roots of $\widetilde{\phi}(t)$ that are roots of unity gives an upper bound for $\rho(\overline{S}_3)$. Using `Magma`, we compute

$$\widetilde{\phi}(t) = \frac{1}{3}(t-1)^2(3t^{20}+t^{19}+t^{17}+t^{16}+2t^{15}+3t^{14}+t^{12}+3t^{11}+2t^{10}+3t^9+t^8+3t^6+2t^5+t^4+t^3+t+3)$$

The roots of the degree 20 factor of $\widetilde{\phi}(t)$ are not integral, and hence they are not roots of unity. We conclude that $\rho(\overline{S}_3) \leq 2$. By [28], we have $\rho(\overline{S}) \leq \rho(\overline{S}_3)$, so $\rho(\overline{S}) \leq 2$. It follows that $S$ (and $\overline{S}$) has Picard rank 2. This concludes the proof of part *d)*.

Finally, the nontriviality of the Clifford invariant follows from Propositions 6 and 7. □

A satisfying feature of Theorem 14 is that we can write out a representative of the Clifford invariant of $X$ explicitly, as a quaternion algebra over the function field of the K3 surface $S$. We first prove a handy lemma, of independent interest for its arithmetic applications (see e.g., [11, 12]).

**Lemma 15.** *Let $K$ be a field of characteristic $\neq 2$ and $q$ a nondegenerate quadratic form of rank 4 over $K$ with discriminant extension $L/K$. For $1 \leq r \leq 4$ denote by $m_r$ the determinant of the leading principal $r \times r$ minor of the symmetric Gram matrix of $q$. Then the class $\beta \in \mathrm{Br}(L)$ of the even Clifford algebra of $q$ is the quaternion algebra $(-m_2, -m_1m_3)$.*

*Proof.* On $n \times n$ matrices $M$ over $K$, symmetric gaussian elimination is the following operation:

$$M = \begin{pmatrix} a & v^t \\ v & A \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & A - a^{-1}vv^t \end{pmatrix}$$

where $a \in K^\times$, $v \in K^{n-1}$ is a column vector, and $A$ is an $(n-1)\times(n-1)$ matrix over $K$. Then $m_1 = a$ and the element in the first row and column of $A - a^{-1}vv^t$ is precisely $m_2/m_1$. By induction, $M$ can be diagonalized, using symmetric gaussian elimination, to the matrix

$$\operatorname{diag}(m_1, m_2/m_1, \ldots, m_n/m_{n-1}).$$

For $q$ of rank 4 with symmetric Gram matrix $M$, we have

$$q = \langle m_1 \rangle \otimes \langle 1, m_2, m_1m_2m_3, m_1m_3m_4 \rangle$$

so that over $L = K(\sqrt{m_4})$, we have that $q \otimes_K L = \langle m_1 \rangle \otimes \langle 1, m_2, m_1m_3, m_1m_2m_3 \rangle$, which is similar to the norm form of the quaternion $L$-algebra with symbol $(-m_2, -m_1m_3)$. Thus the even Clifford algebra of $q$ is Brauer equivalent to $(-m_2, -m_1m_3)$ over $L$. □

**Proposition 16.** *The Clifford invariant of the fourfold $X$ of Theorem 14 is represented by the unramified quaternion algebra $(b, ac)$ over the function field of associated K3 surface $S$, where*

$$a = x - 4y - z, \quad b = x^2 + 14xy - 23y^2 - 8yz,$$

*and*

$$c = 3x^3 + 2x^2y - 4x^2z + 8xyz + 3xz^2 - 16y^3 - 11y^2z - 8yz^2 - z^3.$$

*Proof.* The symmetric Gram matrix of the quadratic form $(\mathscr{O}^3_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-1), q, \mathscr{O}_{\mathbb{P}^2}(1))$ of rank 4 over $\mathbb{P}^2$ associated to the quadric bundle $\pi : \widetilde{X} \to \mathbb{P}^2$ is

$$\begin{pmatrix} 2(x-4y-z) & -x-3y & x-3y & 2x^2+xz-4y^2+2z^2 \\ & 2(-2y) & x-2y-z & x^2-xy-3y^2+yz-z^2 \\ & & 2x & 2x^2+xy+3xz-3y^2+yz \\ & & & 2(x^3+x^2y+2x^2z-xy^2+xz^2-y^3+yz^2-z^3) \end{pmatrix}$$

see [12, §4.2] or [20, §4]. Since $S$ is regular, $\mathrm{Br}(S) \to \mathrm{Br}(k(S))$ is injective; see [2] or [8, Cor. 1.10]. By functoriality of the Clifford algebra, the generic fiber $\beta \otimes_S k(S) \in \mathrm{Br}(k(S))$ is represented by the even Clifford algebra of the generic fiber $q \otimes_{\mathbb{P}^2} k(\mathbb{P}^2)$. Thus we can perform our calculations in the function field $k(S)$. In the notation of Lemma 15, we have $m_1 = 2a$, $m_2 = -b$, and $m_3 = -2c$, and the formulas follow immediately. □

*Remark* 17. Contrary to the situation in [12], the transcendental Brauer class $\beta \in \mathrm{Br}(S)$ is *constant* when evaluated on $S(\mathbb{Q})$; this suggests that arithmetic invariants do not suffice to witness the nontriviality of $\beta$. Indeed, using elimination theory, we find that the odd primes $p$ of bad reduction of $S$ are 5, 23, 263, 509, 1117, 6691, 3342589, 197362715625311, and 4027093318108984867401313726363. For each odd prime $p$ of bad reduction, we compute that the singular locus of $\overline{S}_p$ consists of a single ordinary double point. Thus by [11, Prop. 4.1, Lemma 4.2], the local invariant map associated to $\beta$ is constant on $S(\mathbb{Q}_p)$, for all odd primes $p$ of bad reduction. By an adaptation of [11, Lemma 4.4], the local invariant map is also constant for odd primes of good reduction.

At the real place, we prove that $S(\mathbb{R})$ is connected, hence the local invariant map is constant. To this end, recall that the set of real points of a smooth hypersurface of even degree in $\mathbb{P}^2(\mathbb{R})$ consists of a disjoint union of *ovals* (i.e., topological circles, each of whose complement is homeomorphic to a union of a disk and a Möbius band, in the language of real algebraic geometry). In particular, $\mathbb{P}^2(\mathbb{R}) \smallsetminus D(\mathbb{R})$ has a unique nonorientable connected component $R$. By graphing an affine chart of $D(\mathbb{R})$, we find that the point $(1:0:0)$ is contained in $R$. We compute that the map projecting from $(1:0:0)$ has four real critical values, hence $D(\mathbb{R})$ consists of two ovals. These ovals are not nested, as can be seen by inspecting the graph of $D(\mathbb{R})$ in an affine chart. The Gram matrix of the quadratic form, specialized at $(1:0:0)$, has positive determinant, hence by local constancy, the equation for $D$ is positive over the entire component $R$ and negative over the interiors of the two ovals (since $D$ is smooth). In particular, the map $f : S(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ has empty fibers over the interiors of the two ovals and nonempty fibers over $R \subset \mathbb{P}^2(\mathbb{R})$ where it restricts to a nonsplit unramified cover of degree 2, which must be the orientation double cover of $R$ since $S(\mathbb{R})$ is orientable (the Kähler form on $S$ defines an orientation). In particular, $S(\mathbb{R})$ is connected.

This shows that $\beta$ is constant on $S(\mathbb{Q})$. We believe that the local invariant map is also constant at the prime 2, though this must be checked with a brute force computation.

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