DISTINGUISHING BRILL–NOETHER LOCI

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Abstract. We construct curves carrying certain special linear series and not others, showing many non-containments between Brill–Noether loci in the moduli space of curves. In particular, we prove the Maximal Brill–Noether Loci conjecture in full generality.

Introduction

Whereas classical Brill–Noether theory studies linear systems on general algebraic curves, *refined Brill–Noether theory* aims to characterize linear systems on special curves. A degree d linear system of dimension r, called a g^r_d, corresponds to a non-degenerate morphism $C \to \mathbb{P}^r$ of degree d when it is base point free. The Brill–Noether–Petri theorem [19, 22, 30] states that a general smooth curve C of genus g admits a g^r_d if and only if the *Brill–Noether number*

$$\rho(g,r,d) = g - (r+1)(g-d+r)$$

is non-negative. The last few years have seen a flurry of results concerning refined Brill–Noether theory of curves in a fixed *Brill–Noether locus*

$$\mathcal{M}^r_{g,d} = \{C \in \mathcal{M}_g \ : \ C \text{ admits a } g^r_d\}$$

when $\rho(g,r,d) < 0$. In particular, major advances in refined Brill–Noether theory for curves of fixed gonality, i.e., when $r = 1$, have been made in [11, 25, 28, 29, 39].

Part of a refined Brill–Noether theory concerns the question of whether a "general" curve with a g^r_d carries another $g^{r'}_{d'}$, which can be reinterpreted in terms of containments of Brill–Noether loci. For example, Clifford's theorem implies that $\mathcal{M}^r_{g,2r} \subset \mathcal{M}^1_{g,2}$ for every $r \geq 1$. By adding base points and subtracting non-base points, one obtains the *trivial containments* $\mathcal{M}^r_{g,d} \subset \mathcal{M}^r_{g,d+1}$ and $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$ between Brill–Noether loci. The *expected maximal* Brill–Noether loci are those that do not admit further trivial containments, see Definition 1.1 for the precise definition.

The interaction between various Brill–Noether loci is useful in the study of the birational geometry of $\mathcal{M}_g$, see [17, 24]. When $\rho = -1$, the Brill–Noether loci are irreducible divisors, which have been studied by Harris, Mumford, Eisenbud, and Farkas [14, 15, 17, 18, 24]. A crucial ingredient in the study of the Kodaira dimension of $\mathcal{M}_{23}$ was the maximality of the Brill–Noether divisors.

Inspired by the lifting of line bundles on curves on K3 surfaces, the Donagi–Morrison conjecture, building on work of Farkas and Lelli-Chiesa [17, 18, 31], and classical results in Brill–Noether theory, the first two authors [2] formulated a conjecture stating that the expected maximal Brill–Noether loci are exactly the maximal ones with respect to containment, except precisely in genus $g = 7, 8, 9$, where there are exceptional cases, see Conjecture 1.2, and for details on the exceptional genera see Example 1.3 below. In other words, given any two expected maximal Brill–Noether loci $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$, there is a curve $C \in \mathcal{M}_g$ admitting a g^r_d but not a $g^{r'}_{d'}$. An appealing aspect of the conjecture is a numerical characterization of the maximal elements of the Brill–Noether stratification of $\mathcal{M}_g$. There has been recent progress on this conjecture, referred to as the Maximal Brill–Noether Loci conjecture, in work of many authors [3, 6, 7, 9, 10, 31, 44], which together show the conjecture holds in genus $g \leq 23$ and for infinitely many values of g.

In this paper, we settle the conjecture in the affirmative. In fact, we prove something stronger.

Theorem 1. *In every genus $g \geq 3$, with $g \neq 7, 8, 9$, the expected maximal Brill–Noether loci $\mathcal{M}^r_{g,d}$ are all distinct and are the Brill–Noether loci that are maximal with respect to containment. More precisely, $\mathcal{M}^r_{g,d}$ has a component on which the general curve does not admit any $g^{r'}_{d'}$ with $d' \leq g-1$, $\rho(g,r',d') < 0$, and $(r',d') \neq (r,d)$, unless $(g,r,d) = (7,2,6), (8,1,4), (9,2,7)$.*

We note that this theorem is stronger than the Maximal Brill–Noether Loci conjecture, as it shows the existence of a component in each expected maximal $\mathcal{M}^r_{g,d}$ demonstrating all non-containments.

The curves in Theorem 1 behave as generally as possible for curves admitting a g^r_d. Capturing this idea, we say that a curve $C \in \mathcal{M}^r_{g,d}$ is *g^r_d-general* if any additional Brill–Noether special divisors on C are those determined by Serre duality or by the trivial containments of Brill–Noether loci, see Definition 1.4. We are naturally led to the following question generalizing the Maximal Brill–Noether Loci conjecture.

Question 1. *Does $\mathcal{M}^r_{g,d}$ contain a g^r_d-general curve?*

Theorem 1 gives a positive answer for all expected maximal Brill–Noether loci, unless $g = 7, 8, 9$. Our more general results, see Theorems 2.5 and 5.1, can be used to address additional cases. Previously, special cases of Question 1 have been investigated, see [18]. For $r = 1$, the refined Brill–Noether theory for curves of fixed gonality answers this question, see e.g., [2, Proposition 1.6]. Classically, the genus-degree formula for plane curves gives many examples where $\mathcal{M}^2_{g,d}$ does not contain g^2_d-general curves. More generally, if C can be embedded as a degree d curve in $\mathbb{P}^r$ and admits a k-secant l-plane, then C also carries a g^{r-l-1}_{d-k}, which, if Brill–Noether special, shows that C is not g^r_d-general. Thus Theorem 1 shows that curves in maximal Brill–Noether loci do not admit unexpected secants, cf. [2, Remark 6.5].

Corollary 1. *In every genus $g \geq 3$, with $g \neq 7, 8, 9$, each expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$ with $r \geq 3$ has a component for which a general curve can be embedded as a degree d curve in $\mathbb{P}^r$ that does not admit any k-secant l-plane with $k - (k - l - 1)(r - l) < 0$.*

The main achievement of this paper is to construct smooth curves of genus g carrying a special g^r_d and no $g^{r'}_{d'}$ under certain numerical conditions on the integers g, r, d, r', d'. Our main result is Theorem 5.1 stating that we can construct such curves on K3 surfaces with the g^r_d being induced by a line bundle on the surface. These K3 surfaces live in a divisor $\mathcal{K}^r_{g,d}$ in the moduli space $\mathcal{K}_g$ of quasi-polarized K3 surface of genus g, see §1.2. Despite the apparent complexity of the numerical conditions in Theorem 5.1, it provides a very efficient tool to prove non-containments of Brill–Noether loci in a large number of new cases, in particular, allowing us to deduce Theorem 1. This proof, while inspired by the program initiated in [2], does not rely on any of the special cases of the Maximal Brill–Noether Loci conjecture treated in the literature so far, and in particular, shows that curves on K3 surfaces suffice to distinguish the maximal Brill–Noether loci.

We briefly explain the main ideas behind the proof of Theorem 5.1. In his famous proof of the Brill–Noether–Petri theorem in [30], Lazarsfeld showed that a Brill–Noether special curve on a K3 surface degenerates to a reducible curve, due to the non-simplicity of the associated Lazarsfeld–Mukai bundle. For many (g, r, d), we prove, in Proposition 3.4, that for the very general element $(S, H) \in \mathcal{K}^g_{r,d}$, smooth curves $C \in |H|$ degenerate to a reducible curve $C_1 \cup C_2$ in an essentially unique way, a property we call *decomposition rigidity*, cf. Definition 2.3. We prove that along this unique degeneration, the limit of any Brill–Noether special divisor on C is highly constrained, allowing us to rule out the existence of additional Brill–Noether special divisors. More precisely, the genera of the components C_1 and C_2 are r and $g + r - d - 1$, respectively, and any Brill–Noether special divisor $g^{r'}_{d'}$ on C has "limits" $g^{r_1}_{d_1}$ and $g^{r_2}_{d_2}$ that are Brill–Noether general on the two components, see Theorem 2.5. This imposes numerical restrictions on r_i and d_i, and thus on g, r, d, r', d'. In fact, we show that we can recover a $g^{r'}_{d'}$ on C if the numerical constraints are satisfied, so that Theorem 2.5 gives necessary and sufficient conditions for the existence of a $g^{r'}_{d'}$ on a curve in $|C|$. In particular, this can be interpreted as a "regeneration theorem" for Brill–Noether special linear systems on curves on K3 surfaces.

Outline. In §1, we recall the definitions of expected maximal Brill–Noether loci, the history of progress on distinguishing Brill–Noether loci, as well as background on K3 surfaces and Lazarsfeld–Mukai bundles. In §2, we introduce the notion of decomposition rigidity, and then prove our main technical result, Theorem 2.5, in which we study the limiting behavior of Brill–Noether special divisors on curves on K3 surfaces satisfying decomposition rigidity. In §3, we give sufficient conditions for K3 surfaces in $\mathcal{K}^r_{g,d}$ to satisfy decomposition rigidity, which hold for those associated to expected maximal Brill–Noether loci. In §4, we study the numerical conditions forced on limits of Brill–Noether special divisors. Finally, in §5, we prove our main result, Theorem 5.1, and deduce Theorem 1. We conclude §5 with additional new applications of Theorem 5.1 to non-maximal Brill–Noether loci.

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1. Background on Brill–Noether loci and K3 surfaces

1.1. **Brill–Noether loci.** Surprisingly little is known about the geometry of Brill–Noether loci in general. The expected codimension of $\mathcal{M}^r_{g,d}$ in $\mathcal{M}_g$ is $-\rho(g,r,d)$, and it is known that the codimension of any component of $\mathcal{M}^r_{g,d}$ is at most $-\rho(g,r,d)$. More is known about the existence of components of expected dimension, see for example [7, 27, 40, 42, 44]. However, equidimensionality of components is only known when $-3 \leq \rho(g,r,d) \leq -1$ (additionally assuming $g \geq 12$ when $\rho(g,r,d) = -3$) [13, 43]. Complicating the picture, components of larger than expected dimension can exist, examples include Castelnuovo curves, see for example [40, Remark 1.4]. As mentioned, irreducibility is known for all Brill–Noether loci with $\rho = -1$ by [14] as well as all $\mathcal{M}^2_{g,d}$ with $\rho = -2$ by [9].

There are various containments known among Brill–Noether loci. For example, Clifford's theorem implies that $\mathcal{M}^r_{g,2r} \subset \mathcal{M}^1_{g,2}$. There are *trivial containments* $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^r_{g,d+1}$ obtained by adding a basepoint to a g^r_d on C; and $\mathcal{M}^r_{g,d} \subseteq \mathcal{M}^{r-1}_{g,d-1}$ when $r \geq 2$ obtained by subtracting a non-basepoint, cf. [17, 32]. Modulo these trivial containments, the first two authors defined in [2] the *expected maximal* Brill–Noether loci as follows.

Definition 1.1. *A Brill–Noether locus $\mathcal{M}^r_{g,d}$ is said to be* expected maximal *if $2 \leq d \leq g-1$, $\rho(g,r,d) < 0$, $\rho(g,r,d+1) \geq 0$, and $\rho(g,r-1,d-1) \geq 0$ if $r \geq 2$.*

We will say that a triple (g,r,d) is *associated to an expected maximal Brill–Noether locus* if $\mathcal{M}^r_{g,d}$ is expected maximal.

We remark that, after accounting for Serre duality which gives $\mathcal{M}^r_{g,d} = \mathcal{M}^{g-d+r-1}_{g,2g-2-d}$, every Brill–Noether locus is contained in at least one expected maximal Brill–Noether locus.

The first two authors then posed a conjecture identifying the maximal Brill–Noether loci.

Conjecture 1.2 ([2, Conjecture 1])**.** *In every genus $g \geq 3$, the maximal Brill–Noether loci are the expected maximal loci, except when $g = 7, 8, 9$.*

In other words, the expected maximal Brill–Noether loci should be maximal with respect to containment, except when $g = 7, 8, 9$. In particular, being maximal with respect to containment asks that for every pair of expected maximal Brill–Noether loci $\mathcal{M}^r_{g,d}$ and $\mathcal{M}^{r'}_{g,d'}$, there exists a curve $C \in \mathcal{M}^r_{g,d}$ such that $C \notin \mathcal{M}^{r'}_{g,d'}$. A priori, the curve C could depend on the pair of expected maximal Brill–Noether loci. However, Theorem 1 shows the existence of a component whose general member demonstrates all of the non-containments.

We recall the details of the exceptional cases.

Example 1.3 (Unexpected containments of expected maximal Brill–Noether loci, cf. [2, Props. 6.2-4])**.** We have $\mathcal{M}^2_{7,6} \subset \mathcal{M}^1_{7,4}$, $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$ and $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$, and these containments are strict.

The containment $\mathcal{M}^2_{7,6} \subset \mathcal{M}^1_{7,4}$ follows from the fact that a g^2_6 on a smooth curve of genus 7 cannot be very ample by the genus formula for plane curves, whence the curve must have a g^1_4. The containment $\mathcal{M}^2_{9,7} \subset \mathcal{M}^1_{9,5}$ follows for the same reason. (These were pointed out by H. Larson.)

The containment $\mathcal{M}^1_{8,4} \subset \mathcal{M}^2_{8,7}$ was proven in [34, Lemma 3.8]. Let C be a smooth curve with a g^1_4. Its Serre adjoint is a g^4_{10}. If it is not very ample, C will have a g^3_8, whose Serre adjoint is a g^2_6, whence C carries a g^2_7. If the g^4_{10} is very ample, it embeds C into $\mathbb{P}^4$ as a curve of degree 10, where it has 8 trisecant lines by the Berzolari formula. Hence C has a g^2_7.

Prior to the formulation of Conjecture 1.2 various non-containments between Brill–Noether loci were known. For example, generalizing work of Eisenbud–Harris [15] and Farkas [17] in genus 23, Choi, Kim, and Kim [8, 10, 9] showed that Brill–Noether loci with $\rho = -1, -2$ in every genus have distinct support. Already, this is sufficient to prove Conjecture 1.2 for any genus g such that $g+1$ or $g+2$ is of the form $\operatorname{lcm}(1, 2, \ldots, n)$ for some $n \geq 3$, the first few being $g = 4, 5, 10, 11, 58, 59, 418, 419, 838, 839, 2518, 2519, \ldots$. Choi and Kim [9] showed that Brill–Noether loci with $\rho = -2$ are never contained in each other nor in certain other Brill–Noether divisors. Lelli-Chiesa [31] and the first two authors [2] also showed various non-containments via the lifting of line bundles on curves on K3 surfaces. As conjectured by Pflueger [39] and proved by Jensen–Ranganathan [25] and Cook-Powell–Jensen [11], the combinatorial formula for the expected dimension of Hurwitz–Brill–Noether loci implies that the expected maximal Brill–Noether loci with $r = 1$ are not contained in any other expected maximal loci, except when $g = 8$, see [2, Proposition 1.6]. This result is now part of the full Brill–Noether theory for curves of fixed gonality, established by Larson, Larson, and Vogt [28, 29]. More recently, the gonality stratification of $\mathcal{M}_g$ (work of Larson and the first two authors [3]), strata of differentials (work of Bud [6]), and limit linear series (work of Teixidor i Bigas [44] and Bud and the second author [7]) were also used to show new non-containments between Brill–Noether loci, which all together was sufficient to prove Conjecture 1.2 for $g \leq 23$.

The idea of Conjecture 1.2 is that curves in expected maximal Brill–Noether loci should behave as generally as possible, given that they carry a special linear system. More generally, we are interested in Brill–Noether special curves that admit no further Brill–Noether special divisors, of course allowing those forced by trivial containments of Brill–Noether loci and Serre duality.

Definition 1.4. *We say a curve $C \in \mathcal{M}^r_{g,d}$ is* g^r_d-general *if the only Brill–Noether special divisors $g^{r'}_{d'}$ with $r' \geq 1$ and $d' \leq g-1$ on C are of the form $(r', d') = (r-i, d-i)$ for $0 \leq i \leq r-1$ or $(r', d') = (r, d+j)$ for $0 \leq j \leq g-1-d$.*

In this language, Theorem 1 says that every expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$ contains a g^r_d-general curve, unless $(g, r, d) = (7, 2, 6), (8, 1, 4), (9, 2, 7)$. This leads naturally to Question 1, describing when other Brill–Noether loci contain a g^r_d-general curve. We note that containing a g^r_d-general curve is an open condition in $\mathcal{M}^r_{g,d}$. Various non-trivial containments of Brill–Noether loci give examples of $\mathcal{M}^r_{g,d}$ not containing g^r_d-general curves. The genus-degree formula for plane curves gives containments of the form $\mathcal{M}^2_{g,d} \subseteq \mathcal{M}^1_{g,d-2}$, giving many examples of $\mathcal{M}^2_{g,d}$ not containing g^2_d-general curves. Generalizing this, the existence of a k-secant l-plane to the image of a curve under a g^r_d gives a g^{r-l-1}_{d-k}, which, if Brill–Noether special, gives further examples of curves that are not g^r_d-general. The exceptional cases in Example 1.3 to Conjecture 1.2 are exactly of this form.

The refined Brill–Noether theory for curves of fixed gonality provides a complete answer to Question 1 for $r = 1$. As proved in [25], the general smooth projective k-gonal curve C of genus g

admits a g^r_d if and only if Pflueger's Brill–Noether number

$$\rho_k(g,r,d) \coloneqq \rho(g,r,d) + \max_{0\le \ell \le \min\{r,\ g-d+r+1\}} (g-k-d+2r+1)\ell - \ell^2$$

is non-negative. Hence $\mathcal{M}^1_{g,k}$ contains a g^1_k-general curve if and only if $\rho_k(g,r,d) < 0$ for all (g,r,d) associated to expected maximal Brill–Noether loci with $r \ge 2$.

Example 1.5. The expected maximal locus $\mathcal{M}^1_{g,k}$ has $k = \left\lfloor \frac{g+1}{2} \right\rfloor$, and contains a g^1_k-general curve unless $g = 8$, as shown in [2, Proposition 1.6]. For sub-maximal gonality strata, the situation becomes more complicated. For example, if $k = \left\lfloor \frac{g-1}{2} \right\rfloor \ge 2$, then one can verify that $\mathcal{M}^1_{g,k}$ contains a g^1_k-general curve unless $g = 6, 7, 8, 10, 11, 14$.

We recall the numerical classification of expected maximal Brill–Noether loci from [3].

Lemma 1.6 ([3, Lemma 1.1])**.** *An expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$ exists for some d if and only if*

$$1 \le r \le \begin{cases} \lfloor \sqrt{g} \rfloor & \text{if } g \ge \lfloor \sqrt{g} \rfloor^2 + \lfloor \sqrt{g} \rfloor \\ \lfloor \sqrt{g} \rfloor - 1 & \text{if } g < \lfloor \sqrt{g} \rfloor^2 + \lfloor \sqrt{g} \rfloor . \end{cases} \tag{1}$$

Once a rank r satisfying the conditions of Lemma 1.6 is fixed, the unique degree d that makes $\mathcal{M}^r_{g,d}$ expected maximal is the largest d such that $\rho(g,r,d) < 0$, namely

$$d = d_{max}(g,r) \coloneqq r + \left\lceil \frac{gr}{r+1} \right\rceil - 1. \tag{2}$$

In other words, Lemma 1.6 says that the expected maximal Brill–Noether loci in $\mathcal{M}_g$ are precisely the $\mathcal{M}^r_{g,d}$ for r satisfying (1) and $d = d_{max}(g,r)$.

We remark for later use that an immediate consequence of (1) is the inequality

$$g \ge r^2 + r \tag{3}$$

for (g,r,d) associated to an expected maximal Brill–Noether locus, which can also directly be deduced from the facts that $d \le g-1$ and $\rho(g,r,d+1) \ge 0$.

1.2. **K3 surfaces.** We will work with quasi-polarized K3 surfaces of genus g, that is, with pairs (S,H) where S is a K3 surface and $H \in \mathrm{Pic}(S)$ is a primitive big and nef line bundle such that $H^2 = 2g-2$, so that all smooth irreducible curves in $|H|$ have genus g. To distinguish Brill–Noether loci, we want such curves to carry a g^r_d, which we ensure by specifying the Picard group. In the moduli space $\mathcal{K}_g$ of primitively quasi-polarized K3 surfaces of genus g, the Noether–Lefschetz locus consists of K3 surfaces with Picard rank > 1. Via Hodge theory, the Noether–Lefschetz locus is a union of countably many irreducible Noether–Lefschetz divisors. For $g \ge 2$, $r \ge 0$, and $d \ge 0$, we denote by $\mathcal{K}^r_{g,d}$ the Noether–Lefschetz divisor parameterizing quasi-polarized K3 surfaces $(S,H) \in \mathcal{K}_g$ such that the lattice $\Lambda^r_{g,d} = \mathbb{Z}[H] \oplus \mathbb{Z}[L]$ with intersection matrix

$$\begin{bmatrix} H^2 & H \cdot L \\ L \cdot H & L^2 \end{bmatrix} = \begin{bmatrix} 2g-2 & d \\ d & 2r-2 \end{bmatrix} \tag{4}$$

admits a primitive embedding in $\mathrm{Pic}(S)$ preserving H (using the notation from [2, 23]).

It is well known that $\mathcal{K}^r_{g,d}$ is non-empty if and only if the discriminant

$$\Delta^r_{g,d} \coloneqq 4(g-1)(r-1) - d^2 < 0. \tag{5}$$

Indeed, the Hodge index theorem implies $\Delta^r_{g,d} < 0$ for any $(S,H) \in \mathcal{K}^r_{g,d}$. Surjectivity of the period map (see [33, Thm. 2.9(i)] or [36]) shows there exists a K3 surface S with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$. Acting with *Picard–Lefschetz reflections* on $\Lambda^r_{g,d}$, and using [4, VIII, Prop. 3.9], we may assume that H is nef. In particular, for (g,r,d) satisfying (5), $\mathcal{K}^r_{g,d}$ is non-empty and the very general $(S,H) \in \mathcal{K}^r_{g,d}$ has

$\mathrm{Pic}(S) = \Lambda^r_{g,d}$. We note that when $\mathcal{K}^r_{g,d}$ is non-empty, it is an irreducible divisor of $\mathcal{K}_g$, see [37]. We remark that the $\mathcal{K}^r_{g,d}$ with $\rho(g,r,d) < 0$ are precisely the Noether–Lefschetz divisors parameterizing the Brill–Noether special primitively polarized K3 surfaces in the sense of Mukai [35], see [21, 23].

Up to Serre duality, we can assume that (g,r,d) associated to a Brill–Noether locus satisfies $d \leq g-1$. Hence, for distinguishing Brill–Noether loci, we will focus on K3 surfaces in $\mathcal{K}^r_{g,d}$ satisfying

$$g \geq 3, \ \ r \geq 1, \ \ 2 \leq d \leq g-1. \tag{6}$$

We are interested in $\mathcal{K}^r_{g,d}$ because curves in $|H|$ carry a g^r_d. We make this more precise as follows (cf. [2, Lemma 1.1].

Lemma 1.7. *Let $(S,H) \in \mathcal{K}^r_{g,d}$ and assume that (6) holds. Then any irreducible $C \in |H|$ has genus g and carries a g^r_d, and $|\mathscr{O}_C(L)|$ is a g^r_d if and only if $h^1(L) = h^1(H-L) = 0$.*

Furthermore, if $\mathrm{Pic}(S) \simeq \Lambda^r_{g,d}$, then $|H|$ contains smooth irreducible curves and $|\mathscr{O}_C(L)|$ is a base point free g^r_d for any smooth irreducible $C \in |H|$.

Proof. We note that $L^2 \geq 0$, $(H-L)^2 = 2(g-d-2+r) \geq 2(r-1) \geq 0$, $L \cdot H = d > 0$ and $(H-L)\cdot H = 2(g-1)-d \geq g-1 \geq 2$, whence by Riemann-Roch and Serre duality, L and $H-L$ are both effective and nontrivial. In particular, $h^0(L-H) = 0$ and $\dim |L| = \frac{1}{2}L^2 + 1 + h^1(L) = r + h^1(L) \geq r$. The first assertion in the lemma then follows by adjunction and the sequence

$$0 \longrightarrow \mathscr{O}_S(L-H) \longrightarrow \mathscr{O}_S(L) \longrightarrow \mathscr{O}_C(L) \longrightarrow 0.$$

By classical results [41, Cor. 3.2, § 2.7], a line bundle M on a K3 surface is globally generated if and only if there exists no curve Γ such that $\Gamma^2 = -2$ and $\Gamma.M < 0$ or $\Gamma^2 = 0$ and $\Gamma \cdot M = 1$. Hence, if $\mathrm{Pic}(S) \simeq \Lambda^r_{g,d}$ one may explicitly check, arguing as in e.g. [26], that L and $H-L$ are globally generated, except in one case: $g = 4r-2$, $d = g-1 = 4r-3$ and $H \sim 2D + \Gamma$, with $D \sim L$ or $D \sim H-L$ globally generated and Γ a (-2)-curve such that $D \cdot \Gamma = 1$; in this case, Γ is the base divisor of $|H-D|$. (We leave the details to the reader, since we will not make use of the last assertion in the lemma in this work.) Since $\Gamma \cdot H = 0$, we see that $\mathscr{O}_C(L) \simeq \mathscr{O}_C(H-L) \simeq \mathscr{O}_C(D)$ for any $C \in |H|$. Thus, in all cases, $\mathscr{O}_C(L)$ is base point free.

Since L and $H-L$ are part of a basis of $\mathrm{Pic}(S)$, they cannot be a multiple of an elliptic curve. Therefore, again by classical results [41, Prop. 2.6], L and $H-L$ are represented by an irreducible curve whenever they are globally generated, giving the vanishings $h^1(L) = h^1(H-L) = 0$. In the one remaining case, they have the form $D + \Gamma$ with D represented by an irreducible curve and Γ an irreducible curve intersecting D in one point. Hence, $h^1(D+\Gamma) = 0$ in this case as well, due to 1-connectedness. A similar reasoning shows that H is globally generated and represented by an irreducible curve. Hence, by Bertini and the first part of the lemma, there are smooth irreducible curves $C \in |H|$, and $|\mathscr{O}_C(L)|$ is a g^r_d. □

In the notation of [1, 23], let $\pi : \mathcal{P}_g \to \mathcal{K}_g$ be the universal smooth hyperplane section, whose fiber above (S,H) is the set of smooth irreducible curves in $|H|$, and let $\phi : \mathcal{P}_g \to \mathcal{M}_g$ be the forgetful map. Also denote by $\pi : \mathcal{P}^r_{g,d} \to \mathcal{K}^r_{g,d}$ the restriction to the divisor $\mathcal{K}^r_{g,d} \subset \mathcal{K}_g$. By Lemma 1.7, the image of ϕ restricted to $\mathcal{P}^r_{g,d}$ lies in $\mathcal{M}^r_{g,d}$. We summarize this in the following diagram.

$$\begin{array}{ccc} & \mathcal{P}^r_{g,d} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\phi} \\ \mathcal{K}^r_{g,d} & & \mathcal{M}^r_{g,d} \end{array}$$

Our proof of Theorem 1 will show that when restricted to a general element of $\mathcal{K}^r_{g,d}$, the image of ϕ consists of g^r_d-general curves.

1.3. **Bundles on K3 surfaces.** Let S be a K3 surface and $C \subset S$ a smooth irreducible curve of genus g. A base point free g^r_d on C gives rise to a *Lazarsfeld–Mukai bundle* on S. Denoting by $(\mathscr{L}, V)$ the g^r_d, where $\mathscr{L}$ is a line bundle of degree d on C and $V \subset H^0(\mathscr{L})$ is an $(r+1)$-dimensional subspace, the Lazarsfeld–Mukai bundle $E_{C,(\mathscr{L},V)}$ is defined as the dual of the kernel of the evaluation map $V \otimes \mathscr{O}_S \twoheadrightarrow \mathscr{L}$. We will use the following well-known facts about $E_{C,(\mathscr{L},V)}$, cf. [30].

- $\operatorname{rk}(E_{C,(\mathscr{L},V)}) = r+1$, $c_1(E_{C,(\mathscr{L},V)}) = [C]$, $c_2(E_{C,(\mathscr{L},V)}) = d$;
- $h^2(E_{C,(\mathscr{L},V)}) = 0$;
- $E_{C,(\mathscr{L},V)}$ is globally generated off a finite set;
- $\chi(E^\vee_{C,(\mathscr{L},V)} \otimes E_{C,(\mathscr{L},V)}) = 2(1-\rho(g,r,d))$. In particular, if $\rho(g,r,d) < 0$, then $E_{C,(\mathscr{L},V)}$ is non-simple.

We conclude with a technical lemma which will be useful.

Lemma 1.8. *Let $\mathscr{E}$ be a torsion free coherent sheaf on a K3 surface S that is globally generated off a finite set such that $H^2(\mathscr{E}) = 0$ and $c_1(\mathscr{E}) = E$, where E is a smooth elliptic curve. Then $\operatorname{rk}\mathscr{E} = 1$ and $\mathscr{E}^{\vee\vee} \simeq \mathscr{O}_S(E)$.*

Proof. The cokernel of the canonical inclusion $\mathscr{E} \subset \mathscr{E}^{\vee\vee}$ is supported on a finite set, whence $H^2(\mathscr{E}^{\vee\vee}) = 0$ and $\mathscr{E}^{\vee\vee}$ is globally generated off a finite set. If $\operatorname{rk}\mathscr{E} = 1$, there is nothing to show. If $\operatorname{rk}\mathscr{E} \geq 2$, a general subspace $W \subset H^0(\mathscr{E}^{\vee\vee})$ of $\dim W = \operatorname{rk}\mathscr{E}^{\vee\vee} - 1$ gives rise to an evaluation sequence

$$0 \longrightarrow W \otimes \mathscr{O}_S \longrightarrow \mathscr{E}^{\vee\vee} \longrightarrow \mathscr{O}_S(E) \otimes \mathscr{I}_Z \longrightarrow 0, \tag{7}$$

where Z is a 0-dimensional subscheme of S of length $c_2(\mathscr{E}^{\vee\vee})$. Since $\mathscr{E}^{\vee\vee}$ is globally generated off a finite set we must have $h^0(\mathscr{O}_S(E) \otimes \mathscr{I}_Z) \geq 2$, whence $Z = \emptyset$. As $\mathscr{O}_S(E)$ is globally generated, (7) shows that $\mathscr{E}^{\vee\vee}$ is also. Hence, by a general position argument, for a general subspace $V \subset H^0(\mathscr{E}^{\vee\vee})$ of $\dim V = \operatorname{rk}\mathscr{E}^{\vee\vee}$ the evaluation map $V \otimes \mathscr{O}_S \to \mathscr{E}^{\vee\vee}$ drops rank along a smooth member of $|E|$, which we still denote by E, and we have a short exact sequence

$$0 \longrightarrow V \otimes \mathscr{O}_S \longrightarrow \mathscr{E}^{\vee\vee} \longrightarrow \mathscr{A} \longrightarrow 0, \tag{8}$$

with $\mathscr{A}$ a line bundle on E. Dualizing, we obtain

$$0 \longrightarrow \mathscr{E}^{\vee\vee\vee} \simeq \mathscr{E}^\vee \longrightarrow V^\vee \otimes \mathscr{O}_S \longrightarrow \omega_E \otimes \mathscr{A}^\vee \simeq \mathscr{A}^\vee \longrightarrow 0. \tag{9}$$

The sequences (8) and (9) show that both $\mathscr{A}$ and $\mathscr{A}^\vee$ are globally generated, whence $\mathscr{A} \simeq \mathscr{O}_E$. Since $h^0(\mathscr{E}^\vee) = h^2(\mathscr{E}^{\vee\vee}) = 0$, we get from (9) that $\operatorname{rk}\mathscr{E}^{\vee\vee} = \dim V \leq h^0(\mathscr{O}_E) = 1$, a contradiction. □

2. Brill–Noether special curves on K3 surfaces

The aim of this section is to give a necessary and sufficient criterion to determine whether a curve on a K3 surface with a special linear series induced by a line bundle on the surface can contain other special linear series as well. The main result is summarized in Theorem 2.5 below.

A crucial observation of Lazarsfeld [30, Lemma 1.3] is that a Brill–Noether special curve C on a K3 surface S admits in its complete linear system $|C|$ a reducible curve. More precisely, as will also be clear from the proof of Theorem 2.5, one has an effective decomposition

$$C \sim A + B + T, \tag{10}$$

with A and B globally generated and nontrivial, and T effective (and possibly zero). Control over the (self-)intersection numbers of components of this effective decomposition have been crucial technical inputs into a host of results concerning lifting linear systems to line bundles on K3 surfaces and distinguishing Brill–Noether loci, see [2, 5, 12, 31].

For our purposes, the following will be a convenient definition restricting effective decompositions of H.

Definition 2.1. *Let H be a divisor on a K3 surface S. We say that $H = A + B$ is a* flexible decomposition *if $h^0(A) \geq 2$, $h^0(B) \geq 2$, $A^2 \geq -2$, $B^2 \geq -2$, and at least one of A and B has self-intersection ≥ 0.*

Note that on K3 surfaces containing no (-2)-curves the flexible decompositions are nothing but the effective, nontrivial decompositions.

Example 2.2. For K3 surfaces $(S, H) \in \mathcal{K}^r_{g,d}$ satisfying (6) we have that $H \sim L + (H - L)$ is a flexible decomposition. Indeed, since $(H-L)^2 = 2(g+r-d-2) \geq 0$ and $(H-L)\cdot H = 2g-2-d > 0$, we get by Riemann–Roch that $h^0(H - L) \geq 2$. Similarly for L.

We will henceforth be interested in the uniqueness of flexible decompositions. The following is a weakening of the condition that $|H|$ contains no reducible curves from [30, Theorem].

Definition 2.3. *For a K3 surface S and a divisor H, we say that (S, H) satisfies* decomposition rigidity *if H admits at most one flexible decomposition.*

We will show in Proposition 3.4 that decomposition rigidity is satisfied for general members $(S, H) \in \mathcal{K}^r_{g,d}$ for many (g, r, d), in particular for (g, r, d) associated to expected maximal Brill–Noether loci (cf. Lemma 5.3 and Remark 5.5).

The following lemma, that will be central in the proof of Theorem 2.5, shows the connection between flexible decompositions and the decompositions of Lazarsfeld's form (10).

Lemma 2.4. *Let $(S, H) \in \mathcal{K}^r_{g,d}$ satisfying (6) and decomposition rigidity. If we can write $H \sim A + B + R$, with A, B, R effective and nontrivial, and $A^2 \geq 0$, $B^2 \geq 0$, then either*

(a) *$g + 4r - 2d = 4$ and $r > 1$, or*
(b) *$r = 1$, $g = 2d$, $A = B = L$, $R^2 = -2$, $h^0(R) = 1$ and $L + R$ is nef.*

Proof. As (S, H) satisfies decomposition rigidity, Example 2.2 shows that $H \sim L + (H - L)$ is the unique flexible decomposition. Also note that $h^0(A) \geq 2$ and $h^0(B) \geq 2$.

We first treat the case $R^2 \geq -2$.

We start by proving that

$$R \cdot A \geq -1 \text{ and } R \cdot B \geq -1. \tag{11}$$

Indeed, say $R \cdot A \leq -2$. Then $R^2 = -2$ and nefness of H implies that $R \cdot B \geq 4$ and one easily computes that $(A - jR)^2 \geq 0$ for $j = 1, 2$ and $(B + jR)^2 > 0$ for $j = 1, 2, 3$. Hence,

$$H \sim A + (B + R) \sim (A - R) + (B + R) \sim (A - 2R) + (B + 3R)$$

are all flexible decompositions, contradicting decomposition rigidity. Hence, $R \cdot A \geq -1$, and similarly $R \cdot B \geq -1$, proving (11)

By (11), the two decompositions

$$H \sim (A + R) + B \sim A + (R + B)$$

are flexible, whence decomposition rigidity yields that $A \sim B$, and both are linearly equivalent to L or $H - L$, so that $R \sim H - 2A \sim \pm(H - 2L)$. If $R^2 \geq 0$ or more generally if $h^0(R) \geq 2$, then also $H \sim 2A + R$ is a flexible decomposition, whence decomposition rigidity yields that $R \sim A$, so that $H \sim 3A$, contradicting that primitivity of H. Hence it remains to treat the case $h^0(R) = 1$ and $R^2 = -2$. The latter is equivalent to $g + 4r - 2d = 4$. Hence, if $r > 1$, we are in case (a). If $r = 1$, then $g = 2d$ and we have $H \sim 2A + R$, with $A \sim L$ or $A \sim H - L$. In the latter case we would have $2L \sim H + R$, whence

$$g = 2d = 2L \cdot H = H^2 + R \cdot H = 2g - 2 + R \cdot H,$$

so that $R \cdot H = 2 - g < 0$ (as $g \geq 3$), contradicting nefness of H. Hence, $A \sim L$ and $H \sim 2L + R$, which is case (b), where we have left to prove that $L + R$ is nef. If it were not, there would exist an irreducible (-2)-curve Γ such that $\Gamma \cdot (L+R) < 0$. Since H is nef, we must have $\Gamma \cdot L > 0$ and $\Gamma \cdot R \leq -2$, whence $(R-\Gamma)^2 \geq 0$, yielding the contradiction $h^0(R) \geq h^0(R-\Gamma) \geq 2$.

We next treat the case $R^2 < -2$. Since R is effective, it must contain a (-2)-curve Γ in its support such that $\Gamma \cdot R < 0$, and $R-\Gamma$ is still effective and nontrivial. Since H is nef, we must have $\Gamma \cdot (A+B) \geq 1$, and we can without loss of generality assume that $\Gamma \cdot A \geq 1$. Then $(A+\Gamma)^2 \geq 0$ and we may replace the decomposition $H \sim A+B+R$ with the decomposition $(A+\Gamma)+B+(R-\Gamma)$. Repeating the process we will eventually reach a decomposition of the form $H \sim A'+B'+R'$, with A', B', R' effective and nontrivial, $A'^2 \geq 0$, $B'^2 \geq 0$ and $R'^2 = -2$. This reduces us to the case treated above, which means that we are either in case (a) again, or $r = 1$, $A' \sim B' \sim L$ and $h^0(R') = 1$. We claim that $A \sim A'$ and $B \sim B'$ (so that we end up in case (b) again). Indeed, assume by abuse of notation that at the last but one step of the procedure we have $H \sim A+B+R$, and there is a (-2)-curve Γ such that $\Gamma \cdot R < 0$, $R - \Gamma$ is still effective and nontrivial and $\Gamma \cdot A > 0$. Then $A' \sim A+\Gamma$, $B' \sim B$ and $R' \sim R-\Gamma$. Since $0 = L^2 = A'^2 = (A+\Gamma)^2 = A^2 + 2(\Gamma \cdot A - 1)$, we must have $A^2 = 0$ and $\Gamma \cdot A = 1$. Hence $\Gamma \cdot A' = -1$, so that

$$\Gamma \cdot H = \Gamma \cdot (2A' + R') = \Gamma \cdot (2A' + R - \Gamma) = \Gamma \cdot R < 0,$$

contradicting nefness of H. □

The main result of this section is the following.

Theorem 2.5. *Let $(S, H) \in \mathcal{K}^r_{g,d}$ satisfying* (6) *and decomposition rigidity. Assume further that if $r > 1$, then $g + 4r - 2d \neq 4$. Then $|H|$ contains smooth irreducible curves of genus g, and for any smooth irreducible $C \in |H|$, the linear system $|\mathscr{O}_C(L)|$ is a base point free complete g^r_d, which is very ample if $r \geq 3$. Moreover, there exists a smooth irreducible curve in $|H|$ carrying a $g^{r'}_{d'}$ with $\rho(g, r', d') < 0$ if and only if there exist non-negative integers r_1, r_2, d_1, d_2 such that*

$$r_1 + r_2 = r' - 1 \tag{12}$$

$$d_1 + d_2 \leq d' - d + 2r - 2 \tag{13}$$

$$0 \leq \rho(r, r_1, d_1) < r \tag{14}$$

$$0 \leq \rho(g + r - d - 1, r_2, d_2) < g + r - d - 1. \tag{15}$$

Proof. As (S, H) satisfies decomposition rigidity, Example 2.2 shows that $H \sim L + (H - L)$ is the unique flexible decomposition. Lemma 2.4 and our assumptions imply that L and $H - L$ admit no nontrivial effective decomposition with at least one summand of non-negative self-intersection except in the case $r = 1$, $g = 2d$, $H - L \sim L + R$, with $R^2 = -2$, $h^0(R) = 1$ and $L + R$ nef. Therefore, well-known results on linear systems on K3 surfaces [41, Corollary 3.2, §2.7] imply that L and $H - L$ are globally generated and the general members of $|L|$ and $|H - L|$ are smooth irreducible curves of genus $g(L) = r \geq 1$ and $g(H - L) = g + r - d - 1 \geq 1$, respectively, and that L is even very ample if $r \geq 3$. Similarly, the general member of $|H|$ is a smooth, irreducible curve of genus g. In particular, $h^1(L) = h^1(H - L) = 0$, whence $|\mathscr{O}_C(L)|$ is a base point free complete g^r_d for any smooth $C \in |H|$ by Lemma 1.7, and it is even is very ample if $r \geq 3$ (as L is).

We have left to prove the last statement of the proposition. We first prove the "only if"-part.

Assume that there exists a smooth curve $C \in |H|$ carrying a $g^{r'}_{d'}$ with $\rho(g, r', d') < 0$. Its base point free part is a $g^{r'}_{d''}$ with $d'' \leq d'$. Let $\mathscr{E}$ be the Lazarsfeld–Mukai bundle associated to the base point free part. The fact that $\rho(g, r', d'') \leq \rho(g, r', d') < 0$ implies that $\mathscr{E}$ is non-simple, whence we can by standard arguments find an endomorphism $\varphi : \mathscr{E} \to \mathscr{E}$ dropping rank everywhere (see [30, p. 302]). Set $\mathscr{F} := \operatorname{im} \varphi$ and $\mathscr{G} := \operatorname{coker} \varphi$, which both have positive ranks. Being both a quotient sheaf and subsheaf of $\mathscr{E}$, we have that $\mathscr{F}$ is torsion free, globally generated off a finite set and with $h^2(\mathscr{F}) = 0$. In particular, $\mathscr{F}$ is nontrivial, whence $c_1(\mathscr{F})$ is represented by an effective nonzero divisor F (see, e.g., [30, Fact at the bottom of p. 302]). Since F must be globally generated off a finite set, we must have $F^2 \geq 0$. Let $\mathscr{T}$ be the (possibly zero) torsion subsheaf of $\mathscr{G}$ and $\overline{\mathscr{G}} := \mathscr{G}/\mathscr{T}$. Then $\overline{\mathscr{G}}$ is torsion free, and, being a quotient sheaf of $\mathscr{E}$, it is globally generated off a finite set and satisfies $h^2(\overline{\mathscr{G}}) = 0$. As above, this implies that $c_1(\overline{\mathscr{G}})$ is represented by an effective nonzero divisor G, with $G^2 \geq 0$. We thus have

$$H \sim c_1(\mathscr{E}) \sim c_1(\mathscr{F}) + c_1(\mathscr{G}) \sim c_1(\mathscr{F}) + c_1(\overline{\mathscr{G}}) + c_1(\mathscr{T}) \sim F + G + T,$$

with T an effective (possibly zero) divisor supported on the support of $\mathscr{T}$.

Letting $\overline{\mathscr{F}}$ denote the kernel of the surjection $\mathscr{E} \to \mathscr{G} \to \overline{\mathscr{G}}$, we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \overline{\mathscr{F}} & \longrightarrow & \mathscr{T} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathscr{E} & = & \mathscr{E} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{G} & \longrightarrow & \overline{\mathscr{G}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{16}$$

Note that $\overline{\mathscr{F}}$ is torsion free, being a subsheaf of $\mathscr{E}$. The rightmost vertical sequence in the diagram, together with the facts that $\mathscr{E}$ is locally free and $\overline{\mathscr{G}}$ is torsion free, yields that $\overline{\mathscr{F}}$ is *normal* (cf., e.g., [38, II, Lemma 1.1.16]), whence *reflexive* (cf., e.g., [38, II, Lemma 1.1.12]), whence locally free.

Also note that $h^2(\overline{\mathscr{F}}) = 0$, as $h^2(\mathscr{F}) = 0$.

Consider the standard exact sequence

$$0 \longrightarrow \overline{\mathscr{G}} \longrightarrow \overline{\mathscr{G}}^{\vee\vee} \longrightarrow \tau_{\overline{\mathscr{G}}} \longrightarrow 0, \tag{17}$$

where $\overline{\mathscr{G}}^{\vee\vee}$ is locally free and $\tau_{\overline{\mathscr{G}}}$ has finite support. In particular, also $\overline{\mathscr{G}}^{\vee\vee}$ is globally generated off a finite set and $h^2(\overline{\mathscr{G}}^{\vee\vee}) = 0$. Combining the rightmost vertical sequence in (16) with (17), we obtain

$$0 \longrightarrow \overline{\mathscr{F}} \longrightarrow \mathscr{E} \longrightarrow \overline{\mathscr{G}}^{\vee\vee} \longrightarrow \tau_{\overline{\mathscr{G}}} \longrightarrow 0$$

where $\overline{\mathscr{F}}$ and $\overline{\mathscr{G}}^{\vee\vee}$ are both locally free, $\overline{\mathscr{G}}^{\vee\vee}$ is globally generated off a finite set, $h^2(\overline{\mathscr{F}}) = h^2(\overline{\mathscr{G}}^{\vee\vee}) = 0$, and $\tau_{\overline{\mathscr{G}}}$ supported on a finite set.

Claim 2.6. *Also $\overline{\mathscr{F}}$ is globally generated off a finite set. Moreover, $c_1(\overline{\mathscr{F}})$ and $c_1(\overline{\mathscr{G}}^{\vee\vee})$ are linearly equivalent to L and $H - L$.*

Proof of claim. If $T = 0$, then $\mathscr{T}$ is supported on a finite set. Since $\mathscr{F}$ is also globally generated off finitely many points, the upper horisontal vertical sequence in (16) shows that $\overline{\mathscr{F}}$ is again globally generated off a finite set. Moreover, $c_1(\overline{\mathscr{F}})$ and $c_1(\overline{\mathscr{G}}^{\vee\vee})$ are linearly equivalent to L and $H - L$ by decomposition rigidity.

If $T \neq 0$, then the hypotheses and Lemma 2.4 yield that $r = 1$, $g = 2d$, $F \sim G \sim L$ and $T \sim R \sim H - 2L$, with $R^2 = -2$. In particular, L is represented by a smooth elliptic curve. By Lemma 1.8 we obtain that $\operatorname{rk} \overline{\mathscr{G}}^{\vee\vee} = \operatorname{rk} \mathscr{G} = 1$ and $\operatorname{rk} \overline{\mathscr{F}} = \operatorname{rk} \mathscr{F} = 1$, whence $r' = \operatorname{rk} \mathscr{E} - 1 = 1$. Hence $\overline{\mathscr{G}}^{\vee\vee} \simeq L$ and $\overline{\mathscr{F}} \simeq H - L$. In particular, $\overline{\mathscr{F}}$ is globally generated and the claim follows. □

To simplify notation, we can therefore assume that we have an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{E} \longrightarrow \mathscr{G} \longrightarrow \tau \longrightarrow 0 \tag{18}$$

with $\mathscr{F}$ and $\mathscr{G}$ both locally free of positive ranks and globally generated off finite sets, such that $h^2(\mathscr{F}) = h^2(\mathscr{G}) = 0$ and $c_1(\mathscr{F})$ and $c_1(\mathscr{G})$ are represented by L and $H - L$, and τ is supported on a finite set.

We set $r_{\mathscr{F}} := \operatorname{rk}\mathscr{F} - 1$ and $r_{\mathscr{G}} := \operatorname{rk}\mathscr{G} - 1$, $d_{\mathscr{F}} := c_2(\mathscr{F})$, $d_{\mathscr{G}} := c_2(\mathscr{G})$, $g_{\mathscr{F}} := \frac{1}{2}c_1(\mathscr{F})^2 + 1$ and $g_{\mathscr{G}} := \frac{1}{2}c_1(\mathscr{G})^2 + 1$. Then $r_{\mathscr{F}} \geq 0$ and $r_{\mathscr{G}} \geq 0$. Since $\mathscr{F}$ and $\mathscr{G}$ are globally generated off finite sets, we also have $d_{\mathscr{F}} \geq 0$ and $d_{\mathscr{G}} \geq 0$. Since $\{c_1(\mathscr{F}), c_1(\mathscr{G})\} = \{L, H-L\}$, we have

$$\{g_{\mathscr{F}}, g_{\mathscr{G}}\} = \{g(L), g(H-L)\} = \{r, g+r-d-1\}. \tag{19}$$

From (18) we find

$$r'+1 = \operatorname{rk}\mathscr{E} = \operatorname{rk}\mathscr{F} + \operatorname{rk}\mathscr{G} = r_{\mathscr{F}} + 1 + r_{\mathscr{G}} + 1,$$

that is,

$$r_{\mathscr{F}} + r_{\mathscr{G}} = r' - 1; \tag{20}$$

moreover,

$$\begin{aligned} d' &\geq d'' = c_2(\mathscr{E}) = c_2(\mathscr{F}) + c_2(\mathscr{G}) + c_1(\mathscr{F}) \cdot c_1(\mathscr{G}) + \operatorname{length}\tau \geq d_{\mathscr{F}} + d_{\mathscr{G}} + L \cdot (H-L) \\ &= d_{\mathscr{F}} + d_{\mathscr{G}} + d - 2r + 2, \end{aligned}$$

in other words

$$d_{\mathscr{F}} + d_{\mathscr{G}} \leq d' - d + 2r - 2. \tag{21}$$

If $r_{\mathscr{F}} = 0$, then $\operatorname{rk}\mathscr{F} = 1$, so that $d_{\mathscr{F}} = c_2(\mathscr{F}) = 0$. This implies that $\rho(g_{\mathscr{F}}, r_{\mathscr{F}}, d_{\mathscr{F}}) = \rho(g_{\mathscr{F}}, 0, 0) = 0$. If $r_{\mathscr{F}} > 0$, then, as $\mathscr{F}$ is globally generated off a finite set, we have a short exact sequence

$$0 \longrightarrow \mathbb{C}^{r_{\mathscr{F}}+1} \otimes \mathscr{O}_S \longrightarrow \mathscr{F} \longrightarrow \mathscr{A}_{\mathscr{F}} \longrightarrow 0, \tag{22}$$

with $\mathscr{A}_{\mathscr{F}}$ a torsion free rank-one sheaf on a reduced and irreducible member $D \in |c_1(\mathscr{F})|$. As $h^2(\mathscr{F}) = 0$, the sheaf $\mathscr{F}$ is nontrivial, whence $h^0(\mathscr{F}) > \operatorname{rk}(\mathscr{F}) = r_{\mathscr{F}} + 1$, so that

$$h^0(\mathscr{A}_{\mathscr{F}}) > 0. \tag{23}$$

Dualizing (22), we obtain

$$0 \longrightarrow \mathscr{F}^{\vee} \longrightarrow \mathbb{C}^{r_{\mathscr{F}}+1} \otimes \mathscr{O}_S \longrightarrow \mathscr{B}_{\mathscr{F}} := \mathscr{E}xt^1(\mathscr{A}_{\mathscr{F}}, \mathscr{O}_S) \longrightarrow 0, \tag{24}$$

again with $\mathscr{B}_{\mathscr{F}}$ a torsion free rank-one sheaf on D, which is globally generated by construction and satisfies $\deg(\mathscr{B}_{\mathscr{F}}) = c_2(\mathscr{F}) = d_{\mathscr{F}}$. Since $h^0(\mathscr{F}^{\vee}) = h^2(\mathscr{F}) = 0$, we see that $h^0(\mathscr{B}_{\mathscr{F}}) = r_{\mathscr{F}} + 1 + h^1(\mathscr{F})$. Now choose $h^1(\mathscr{F})$ general points $x_1, \ldots, x_{h^1(\mathscr{F})}$ in the smooth locus of D and set $\mathscr{B}'_{\mathscr{F}} := \mathscr{B}_{\mathscr{F}}(-x_1 - \cdots - x_{h^1(\mathscr{F})})$. Then

$$\begin{aligned} h^0(\mathscr{B}'_{\mathscr{F}}) &= h^0(\mathscr{B}_{\mathscr{F}}) - h^1(\mathscr{F}) = r_{\mathscr{F}} + 1, \\ h^1(\mathscr{B}'_{\mathscr{F}}) &= h^1(\mathscr{B}_{\mathscr{F}}) = h^0(\mathscr{E}xt^1(\mathscr{B}_{\mathscr{F}}, \mathscr{O}_S)) = h^0(\mathscr{A}_{\mathscr{F}}) > 0, \end{aligned}$$

using (23) (cf. [20, Lemmas 2.1 and 2.3]). Moreover, $\mathscr{B}'_{\mathscr{F}}$ is still globally generated, so we have a short exact sequence

$$0 \longrightarrow \mathscr{F}'^{\vee} \longrightarrow \mathbb{C}^{r_{\mathscr{F}}+1} \otimes \mathscr{O}_S \longrightarrow \mathscr{B}'_{\mathscr{F}} \longrightarrow 0, \tag{25}$$

defining a new Lazarsfeld–Mukai bundle $\mathscr{F}'$. We have $\operatorname{rk}(\mathscr{F}') = \operatorname{rk}(\mathscr{F}) = r_{\mathscr{F}} + 1$, $c_1(\mathscr{F}') = c_1(\mathscr{F})$ and $d'_{\mathscr{F}} := c_2(\mathscr{F}') = \deg(\mathscr{B}'_{\mathscr{F}}) = d_{\mathscr{F}} - h^1(\mathscr{F}) \leq d_{\mathscr{F}}$. Dualizing (25) we obtain

$$0 \longrightarrow \mathbb{C}^{r_{\mathscr{F}}+1} \otimes \mathscr{O}_S \longrightarrow \mathscr{F}' \longrightarrow \mathscr{E}xt^1(\mathscr{B}'_{\mathscr{F}}, \mathscr{O}_S) \longrightarrow 0, \tag{26}$$

and since $h^0(\mathscr{E}xt^1(\mathscr{B}'_{\mathscr{F}}, \mathscr{O}_S)) = h^1(\mathscr{B}'_{\mathscr{F}}) > 0$ (cf. [20, Lemma 2.3]), we see that $\mathscr{F}'$ is globally generated off a finite set. Recall that $D \sim L$ or $H - L$, and, as a consequence of what we said in the first lines of the proof, L and $H - L$ admit no decompositions in moving classes. Hence

$$\rho(g_{\mathscr{F}}, r_{\mathscr{F}}, d'_{\mathscr{F}}) \geq 0,$$

as in the proof of [30, Lemma 1.3]. At the same time, since $h^0(\mathscr{B}'_{\mathscr{F}}) > 0$ and $h^1(\mathscr{B}'_{\mathscr{F}}) > 0$, we also have

$$\rho(g_{\mathscr{F}}, r_{\mathscr{F}}, d'_{\mathscr{F}}) = g(D) - h^0(\mathscr{B}'_{\mathscr{F}})h^1(\mathscr{B}'_{\mathscr{F}}) < g(D) = g(\mathscr{F}).$$

To summarize, we have in any event found an integer $d'_{\mathscr{F}}$ such that $0 \leq d'_{\mathscr{F}} \leq d_{\mathscr{F}}$ and

$$0 \leq \rho(g_{\mathscr{F}}, r_{\mathscr{F}}, d'_{\mathscr{F}}) < g_{\mathscr{F}}. \tag{27}$$

Similarly, considering $\mathscr{G}$, we find an integer $d'_{\mathscr{G}}$ such that $0 \leq d'_{\mathscr{G}'} \leq d_{\mathscr{G}}$ and

$$0 \leq \rho(g_{\mathscr{G}}, r_{\mathscr{G}}, d'_{\mathscr{G}}) < g_{\mathscr{G}}. \tag{28}$$

By (21) we see that

$$d'_{\mathscr{F}} + d'_{\mathscr{G}} \leq d' - d + 2r - 2. \tag{29}$$

Recalling (19), we see that (20), (29) and (27)-(28) imply (12), (13) and (14)-(15), with $\{r_1, r_2\} = \{r_{\mathscr{F}}, r_{\mathscr{G}}\}$ and $\{d_1, d_2\} = \{d'_{\mathscr{F}}, d'_{\mathscr{G}}\}$.

We then prove the "if"-part.

Assume the existence of the integers as stated in the theorem. Assume first that $r_1, r_2 > 0$. Then, by (14)-(15), on any smooth curve in $|L|$ and $|H - L|$ there exist a complete special $g^{r_1}_{d_1}$ and $g^{r_2}_{d_2}$, respectively. Denote by $\mathscr{A}_i$, $i = 1, 2$, the corresponding line bundles, and by D_i the corresponding curves. Define $\mathscr{A}'_i$ to be the base point free part of $\mathscr{A}_i$, that is, the image of the evaluation map $H^0(\mathscr{A}_i) \otimes \mathscr{O}_{D_i} \to \mathscr{A}_i$, by $\mathscr{B}_i$ the base point free part of $\omega_{D_i} \otimes {\mathscr{A}'}^{-1}_i$, and set $\overline{\mathscr{A}}_i := \omega_{D_i} \otimes \mathscr{B}_i^{-1}$. Then $h^0(\overline{\mathscr{A}}_i) = h^0(\mathscr{A}_i) + l_i$ and $\deg(\overline{\mathscr{A}}_i) = \deg(\mathscr{A}'_i) + l_i \leq d_i + l_i$, where l_i is the length of the base locus of $|\omega_{D_i} \otimes {\mathscr{A}'}^{-1}_i|$. One easily checks, as e.g. in [20, Lemma 3.1], that both $|\overline{\mathscr{A}}_i|$ and $|\omega_{D_i} \otimes \overline{\mathscr{A}}_i^{-1}|$ are base point free. Hence, we have found, on (all) smooth curves in $|L|$ and $|H - L|$, a base point free complete $g^{r_i+l_i}_{d'_i+l_i}$, with $d'_i \leq d_i$ and $l_i \geq 0$, for $i = 1, 2$, respectively, and such that its adjoint is also base point free.

Let $\mathscr{F}$ and $\mathscr{G}$ be the associated Lazarsfeld–Mukai bundles, of ranks $r_1 + l_1 + 1$ and $r_2 + l_2 + 1$, respectively, which are globally generated by the assumption on the base point freeness of the adjoint linear systems. Set $\mathscr{E} := \mathscr{F} \oplus \mathscr{G}$. Then

$$\mathrm{rk}(\mathscr{E}) = r_1 + l_1 + r_2 + l_2 + 2 = r' + 1 + l_1 + l_2$$

by (12) and

$$\begin{aligned} c_2(\mathscr{E}) &= c_2(\mathscr{F}) + c_2(\mathscr{G}) + c_1(\mathscr{F}) \cdot c_1(\mathscr{G}) = d'_1 + l_1 + d'_2 + l_2 + L \cdot (H - L) \\ &= d'_1 + l_1 + d'_2 + l_2 + d - 2r + 2 \leq d_1 + l_1 + d_2 + l_2 + d - 2r + 2 \\ &\leq d' + l_1 + l_2, \end{aligned}$$

by (13). Then, as is well-known, the evaluation map $\mathbb{C}^{r'+1} \otimes \mathscr{O}_S \to \mathscr{E}$ drops rank along a smooth curve $C \in |H|$ and the cokernel is a line bundle $\mathscr{A}$ on C, such that $|\omega_C - \mathscr{A}|$ is a $g^{r'+l_1+l_2}_{c_2(\mathscr{E})}$; in particular C carries a $g^{r'+l_1+l_2}_{d'+l_1+l_2}$, whence also a $g^{r'}_{d'}$.

Assume now that $r_1 = 0$ and $r_2 > 0$. Then by (15) there exist a complete special $g^{r_2}_{d_2}$ on any smooth curve in $|H - L|$, and as before, we find a complete, base point free $g^{r_2+l_2}_{d'_2+l_2}$ with $d'_2 \leq d_2$ and $l_2 \geq 0$, such that its adjoint is again globally generated. Letting $\mathscr{F}$ be its associated Lazarsfeld–Mukai bundle of rank $r_2 + l_2 + 1$, which again is globally generated, we set $\mathscr{E} := L \oplus \mathscr{F}$, and argue as above. Similarly if $r_2 = 0$ and $r_1 > 0$.

Finally, if $r_1 = r_2 = 0$, we set $\mathscr{E} = L \oplus (H - L)$ and argue as before. □

Remark 2.7. A careful look at the proofs of Lemma 2.4 and Theorem 2.5 shows that the condition "$(S, H) \in \mathcal{K}^r_{g,d}$" can be replaced by "$(S, H)$ a quasi-polarized K3 surface of genus g with a line bundle $L \in \mathrm{Pic}(S)$ satisfying $L^2 = 2r - 2$, $L.H = d$ and $H \not\sim 3L$". Also note that whenever H is n-divisible in $\mathrm{Pic}(S)$ for an $n \geq 4$, then (S, H) does not satisfy decomposition rigidity.

3. Flexible decompositions on K3s of Picard number two

We want to find K3 surfaces satisfying the conditions in Theorem 2.5, in particular decomposition rigidity. Throughout the section we will fix integers (g,r,d) satisfying (5) and (6), and a pair $(S,H) \in \mathcal{K}^r_{g,d}$ with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$. For simplicity we write $\Delta := \Delta^r_{g,d}$.

Assume now that $H \sim A + B$ is a flexible decomposition of H. We may write

$$A = xH - yL \ \text{ and } \ B = (1-x)H + yL \ \text{ for some } \ x, y \in \mathbb{Z}.$$

We may and will assume that $A^2 \geq 0$ and $B^2 \geq -2$. The next two lemmas give necessary conditions on x and y.

Lemma 3.1. *We have*

$$(g-1)x^2 - dxy + (r-1)y^2 \geq 0, \tag{30}$$

$$(g-1)(1-x)^2 + d(1-x)y + (r-1)y^2 \geq -1, \tag{31}$$

$$\tfrac{2(g-1)}{d}(x-1) < y < \tfrac{2(g-1)}{d}x. \tag{32}$$

Proof. The two first conditions are equivalent to $\frac{1}{2}A^2 \geq 0$ and $\frac{1}{2}B^2 \geq -1$, respectively. The last condition is equivalent to $A \cdot H > 0$ and $B \cdot H > 0$, which are satisfied since $h^0(A) \geq 2$ and $h^0(B) \geq 2$ and H is big and nef. □

Lemma 3.2. *If $r = 1$, then*

$$x = 0 \ \text{ and } \ -\frac{g}{d} \leq y < 0, \qquad \text{or } \ x = 1 \ \text{ and } \ 0 < y \leq \frac{g-1}{d}.$$

If $r \geq 2$, then either

$$0 < x \leq \max\left\{1 + \tfrac{d}{\sqrt{|\Delta|}\sqrt{g-1}} \ , \ \tfrac{2(r-1)g}{(d-\sqrt{|\Delta|})\sqrt{|\Delta|}}\right\} \ \text{ and} \tag{33}$$
$$\tfrac{2(g-1)}{d}(x-1) < y \leq \min\left\{\tfrac{d-\sqrt{|\Delta|}}{2(r-1)}x \ , \ \tfrac{g}{\sqrt{|\Delta|}}\right\}$$

or

$$-\tfrac{2(r-1)g}{(d+\sqrt{|\Delta|})\sqrt{|\Delta|}} \leq x \leq 0 \ \text{ and} \tag{34}$$
$$\max\left\{-\tfrac{g}{\sqrt{|\Delta|}} \ , \ \tfrac{2(g-1)}{d}(x-1)\right\} \leq y \leq \tfrac{d+\sqrt{|\Delta|}}{2(r-1)}x, \ \ y \neq 0, \ \ y \neq \tfrac{2(g-1)}{d}(x-1).$$

Proof. We first treat the case $r = 1$.

If $x < 0$, then (30) and the right inequality in (32) yield

$$(g-1)x \leq dy < 2(g-1)x,$$

whence the contradiction $(g-1)x > 0$. If $x > 0$, then (30) and the left inequality in (32) yield

$$2(g-1)(x-1) < dy \leq (g-1)x,$$

which implies $(g-1)x < 2(g-1)$, whence $x = 1$, and thus $0 < y \leq \frac{g-1}{d}$.

If $x = 0$, then (31) and the left inequality in (32) yield $-\frac{g}{d} \leq y < 0$.

The rest of the proof will deal with the case $r \geq 2$.

We first note that (32) implies that

$$\text{either } x \leq 0 \text{ and } y < 0, \text{ or } x > 0 \text{ and } y > 0. \tag{35}$$

Letting $a_\pm := \frac{d \pm \sqrt{|\Delta|}}{2(r-1)}$, we note that (30) factors as

$$(r-1)(y - a_+x)(y - a_-x) \geq 0. \tag{36}$$

Define the following lines in the plane:

$$\ell_+ :\ y = a_+ x, \quad \ell_- :\ y = a_- x, \quad \ell :\ y = \frac{2(g-1)}{d}(x-1).$$

Since $a_+ > \frac{2(g-1)}{d} > a_- > 0$, the conditions (36) and (32) yield, also recalling (35), that either

$$x > 0 \ \text{ and } \ 0 \leq \frac{2(g-1)}{d}(x-1) < y \leq a_- x \tag{37}$$

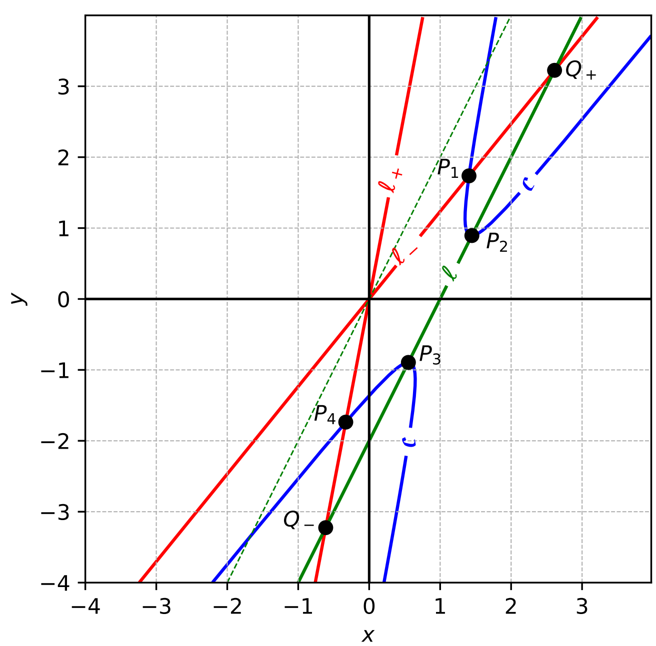
(which determines a triangle bounded by the x-axis and the lines ℓ and ℓ_- in the first quadrant) or

$$x \leq 0 \ \text{ and } \ \frac{2(g-1)}{d}(x-1) < y \leq a_+ x, \ \ y < 0 \tag{38}$$

(which determines a triangle bounded by the y-axis and the lines ℓ and ℓ_+ in the third quadrant). Furthermore, condition (31) implies that the points (x, y) must lie in the region containing the origin determined by the hyperbola with two branches

$$\mathfrak{c} :\ (g-1)(1-x)^2 + d(1-x)y + (r-1)y^2 = -1,$$

as shown in the following picture, which shows the case $(g, r, d) = (14, 3, 13)$.



We compute the intersection points between the curves, as shown in the picture:

$$\begin{aligned}
Q_+ = (x_+, y_+) &\coloneqq \ell \cap \ell_- &=& \left(1 + \frac{d}{\sqrt{|\Delta|}}, \frac{2(g-1)}{\sqrt{|\Delta|}}\right), \\
Q_- = (x_-, y_-) &\coloneqq \ell \cap \ell_- &=& \left(1 - \frac{d}{\sqrt{|\Delta|}}, -\frac{2(g-1)}{\sqrt{|\Delta|}}\right), \\
P_1 = (x_1, y_1) &\coloneqq \mathfrak{c} \cap \ell_- &=& \left(\frac{2(r-1)g}{(d-\sqrt{|\Delta|})\sqrt{|\Delta|}} \ , \ \frac{g}{\sqrt{|\Delta|}}\right), \\
P_4 = (x_4, y_4) &\coloneqq \mathfrak{c} \cap \ell_+ &=& \left(-\frac{2(r-1)g}{(d+\sqrt{|\Delta|})\sqrt{|\Delta|}} \ , \ -\frac{g}{\sqrt{|\Delta|}}\right), \\
P_2 = (x_2, y_2) &\coloneqq \mathfrak{c} \cap \ell \cap \{y \geq 0\} &=& \left(1 + \frac{d}{\sqrt{g-1}\sqrt{|\Delta|}} \ , \ \frac{2\sqrt{g-1}}{\sqrt{|\Delta|}}\right), \\
P_3 = (x_3, y_3) &\coloneqq \mathfrak{c} \cap \ell \cap \{y \leq 0\} &=& \left(1 - \frac{d}{\sqrt{g-1}\sqrt{|\Delta|}} \ , \ -\frac{2\sqrt{g-1}}{\sqrt{|\Delta|}}\right).
\end{aligned}$$

We first consider case (37). We note that $x_2 \leq x_+$ and $y_+ \geq y_1 \geq y_2$. Thus, the points (x, y) lie in the region bounded by the x-axis, the lines ℓ_- and ℓ, and the segment of $\mathfrak{c}$ between P_1 and P_2. Since $\mathfrak{c}$ is convex, the points (x, y) are contained in the pentagon bounded by the x-axis, the lines ℓ_- and ℓ, and the line segment P_1P_2. In particular, $x \leq \max\{x_1, x_2\}$ and $y \leq y_1$. Together with (37), this yields (33).

We next consider case (38). We note that $x_- \leq x_3$ and $y_- \leq y_4 \leq y_3$. The latter implies that the intersection points P_4 and P_3 occur in the region where ℓ_+ lies to the left of ℓ, whence it follows that also $x_4 \leq x_3$. (Note, however, that contrary to the impression one may get by the picture, we could have $x_3 < 0$.) The points (x, y) lie in the region bounded by the y-axis, the lines ℓ_+ and ℓ and the the segment of $\mathfrak{c}$ between P_4 and P_3. Since $\mathfrak{c}$ is convex, the points (x, y) are contained in the pentagon bounded by the y-axis, the lines ℓ_+ and ℓ, and the line segment P_4P_3. In particular, $x \geq x_4$ and $y \leq y_4$. Together with (38), this yields (34). □

Remark 3.3. A direct computation shows that

$$\frac{2(r-1)g}{(d-\sqrt{|\Delta|})\sqrt{|\Delta|}} - \frac{2(r-1)g}{(d+\sqrt{|\Delta|})\sqrt{|\Delta|}} = 1 + \frac{1}{g-1}.$$

In particular, setting $m \coloneqq \left\lfloor \max\left\{1 + \frac{d}{\sqrt{|\Delta|}\sqrt{g-1}} \ , \ \frac{g(2r-2)}{(d-\sqrt{|\Delta|})\sqrt{|\Delta|}}\right\}\right\rfloor$, Lemma 3.2 shows that $x \in \{-m+1, \ldots, m\}$.

As a consequence, we can deduce decomposition rigidity under certain assumptions.

Proposition 3.4. *Assume that* (6) *holds,*

$$d \geq \begin{cases} \frac{g-3}{2} + 2r, & \textit{if } r \geq 2, \\ \frac{g}{2}, & \textit{if } r = 1, \end{cases} \tag{39}$$

and

$$(r-1)(3g-4)^2 < 2d^2(g-2). \tag{40}$$

Then any $(S, H) \in \mathcal{K}^r_{g,d}$ *with* $\operatorname{Pic}(S) = \Lambda^r_{g,d}$ *satisfies decomposition rigidity.*

Remark 3.5. Condition (39) together with the assumption that $d \leq g-1$ (cf. (6)) imply that $g + 1 \geq 4r$.

Remark 3.6. If (39) is not fulfilled, then $(H-2L)^2 \geq \begin{cases} -2, & \text{if } r \geq 2, \\ 0, & \text{if } r=1, \end{cases}$ whence either $H \sim 2L+(H-2L)$ is another flexible decomposition, or the assumption in Theorem 2.5 that $g+4r-2d \neq 4$, which is equivalent to $(H-2L)^2 \neq -2$, is not satisfied. Thus (39) is the optimal bound to be able to apply Theorem 2.5.

Remark 3.7. Condition (40) is automatically fulfilled if $r=1$. Moreover, under assumption (39), condition (40) is for instance implied by the simplified assumption

$$g \geq \min\{r(r+1),\ 10(r-1)\} = \begin{cases} r(r+1), & \text{if } 2 \leq r \leq 7, \\ \\ 10(r-1), & \text{if } r \geq 8. \end{cases} \tag{41}$$

Indeed, because of (39), condition (40) is implied by

$$(r-1)(3g-4)^2 < 2(g-2)\left(\frac{g-3}{2}+2r\right)^2,$$

equivalently

$$g^3+g^2(-10r+10)+g(16r^2+8r-27)-32r^2+16r+14>0. \tag{42}$$

As $g(16r^2+8r-27)-32r^2+16r+14>0$ for $r \geq 2$ and $g \geq 3$, we see that (42) holds if $g \geq 10(r-1)$. If $r \leq 7$, one can check that (42) holds when $g \geq r(r+1)$. Hence (40) holds.

Proof of Proposition 3.4. We want to prove that $H \sim L+(H-L)$ is the unique flexible decomposition of H. To this aim, let $H \sim A+B$ where $A = xH-yL$ and $B=(1-x)H+yL$ be a flexible decomposition of H. We will prove that $(x,y)=(1,1)$ or $(0,-1)$, which both correspond to the decomposition $H \sim (H-L)+L$.

If $r=1$, assumption (39) and Lemma 3.2 show that the only possibilities are $(x,y)=(1,1)$, $(0,-1)$ or $(0,-2)$. In the latter case we have $B \sim H-2L$ and $B^2=-2$. We claim that B is the only (-2)-divisor on S, which will prove that B is irreducible, giving the contradiction $h^0(B)=1$. Indeed, if Γ is a (-2)-divisor on S, we may write $\Gamma \sim aH-bL$, with $a>0$, since multiples of L can never contain H. Then $-1=\frac{1}{2}\Gamma^2 = a[a(g-1)-bd]$, whence $a=1$ and $g=bd$, so $b=2$ and $\Gamma \sim H-2L \sim B$. This gives the desired contradiction, showing that the case $(x,y)=(0,-2)$ does not occur, as desired.

Now suppose $r \geq 2$.

We first claim that if $-3 \leq y \leq 3$ and $0 \leq x \leq 1$, then again $(x,y)=(1,1)$ or $(0,-1)$. Indeed, Lemma 3.2 shows that it suffices to rule out the cases $(x,y)=(0,-2),(0,-3),(1,2)$ and $(1,3)$. In these cases one of A or B is $H-2L$ or $H-3L$. By (39) we have

$$(H-2L)^2 = 2g+8r-10-4d \leq 2g+8r-10-4\left(\frac{g-3}{2}+2r\right) = -4,$$

a contradiction. Similarly, we have

$$-2 \leq (H-3L)^2 = 2g+18r-20-6d \leq 2g+18r-20-6\left(\frac{g-3}{2}+2r\right) = 6r-11-g,$$

whence $g \leq 6r-9$. But then, using this and (39) once more, we obtain the contradiction

$$(H-3L)\cdot H = 2g-2-3d \leq 2g-2-3\left(\frac{g-3}{2}+2r\right) = \frac{g+5}{2}-6r \leq \frac{1}{2}(6r-4)-6r = -3r-2<0.$$

It remains to show that $-3 \leq y \leq 3$ and $0 \leq x \leq 1$. By Remark 3.3 and Lemma 3.2, to show that $0 \leq x \leq 1$ it suffices to show that

$$1+\frac{d}{\sqrt{|\Delta|}\sqrt{g-1}} < 2 \tag{43}$$

and

$$\frac{2(r-1)g}{(d-\sqrt{|\Delta|})\sqrt{|\Delta|}} < 2. \tag{44}$$

Having done this, the case $x = 1$ yields, because of the right inequality in (32), that

$$0 < y < \frac{2(g-1)}{d} \leq \frac{2(g-1)}{\frac{g-3}{2} + 2r} \leq \frac{4(g-1)}{g+5} < 4,$$

and similarly the case $x = 0$ yields, because of the left inequality in (32), that $0 < -y < 4$. We have therefore left to prove (43) and (44).

Inequality (44) is equivalent to

$$d^2 - (r-1)(3g-4) < d\sqrt{|\Delta|}.$$

The latter is implied by the inequality obtained by squaring both sides, which is (40). Thus, (44) is satisfied.

Inequality (43) is equivalent to

$$4(g-1)^2(r-1) < (g-2)d^2. \tag{45}$$

Our assumptions yield that $g \geq 4r - 1 \geq 7$, cf. Remark 3.5, and then one readily checks that $4(g-1)^2 \leq \frac{1}{2}\left(3g-4\right)^2$, so that (45) is implied by (40). Thus, (43) is also satisfied. □

4. Bounds on the limits of Brill–Noether special divisors

The aim of this section is to give sufficient conditions so that (12)-(15) in Theorem 2.5 cannot be verified, thus ruling out the existence of a $g^{r'}_{d'}$ as in that theorem.

Lemma 4.1. *Assume that $r' \geq 1$, $2 \leq d' \leq g-1$, (6) is satisfied, and that*

$$\begin{cases} d - r + r' - d' > \frac{r}{r'} - 1, & \text{if } r' \leq r, \\ g + r' - 1 - d' - \frac{g+r-d-1}{r'-r+1} > 0, & \text{if } r < r'. \end{cases} \tag{46}$$

Then there are no non-negative integers r_1, r_2, d_1, d_2 satisfying (12)–(15).

Proof. Assume, to get a contradiction, that r_1, r_2, d_1, d_2 are non-negative integers satisfying (12)–(15).

We have

$$0 \leq r_1 \leq r' - 1. \tag{47}$$

Condition (14) can be rewritten as

$$r_1 + \frac{r_1 r}{r_1 + 1} \leq d_1 < r_1 + r. \tag{48}$$

From the left hand inequality of (15) combined with (12) and (13), along with the fact that $\rho(g, r, -)$ is an increasing function, we obtain

$$\rho(g + r - d - 1, r' - 1 - r_1, d' - d + 2r - 2 - d_1) \geq 0,$$

which can be rewritten as

$$d_1 \leq d' + r + r_1 - r' - g + \frac{g + r - d - 1}{r' - r_1}. \tag{49}$$

Grant for the moment the following:

Claim 4.2. *If*

- $r' \leq r$, *or*
- $r_1 < r < r'$,

then

$$d' + r + r_1 - r' - g + \frac{g+r-d-1}{r'-r_1} < r_1 + \frac{r_1 r}{r_1+1}. \tag{50}$$

We show how the lemma follows from the claim.

Assume that $r' \leq r$. Then Claim 4.2 shows that (49) and the left hand inequality of (48) are incompatible. We are therefore done in this case.

Assume, finally, that $r' > r$. Then Claim 4.2 shows that (49) and the left hand inequality of (48) are compatible only for $r_1 > r-1$. This latter inequality is on the other hand equivalent to

$$r + r_1 - \left(r_1 + \frac{rr_1}{r_1+1}\right) < 1.$$

Thus there is no integer d_1 satisfying (48). This concludes the proof of the lemma.

We have left to prove the claim:

Proof of Claim 4.2. Since the left hand side of (50) is a convex function of r_1 and the right hand side is a concave function of r_1, it suffices to prove the inequality for the two boundary values $r_1 = 0$ and

$$r_1 = \begin{cases} r'-1, & \text{if } r' \leq r, \\ r-1 & \text{if } r' > r, \end{cases}$$

cf. (47), where it reads, respectively,

$$d' + r - r' - g + \frac{g+r-d-1}{r'} < 0, \tag{51}$$

$$d' - d + 2r - 2 < r' - 1 + \frac{(r'-1)r}{r'}, \text{ if } r' \leq r, \tag{52}$$

and

$$d' + 1 - r' - g + \frac{g+r-d-1}{r'-r+1} < 0, \text{ if } r' > r. \tag{53}$$

One easily verifies that (52) is equivalent to the upper inequality in (46) and implies (51). We are therefore done in the case $r' \leq r$.

Assume now that $r' > r$. If $r \geq r' - \frac{g-d-1}{r'+1}$, one verifies that (53) implies (51). If $r < r' - \frac{g-d-1}{r'+1}$, then $g-d-1 < (r'-r)(r'+1)$. Using this together with $d' \leq g-1$, one checks that (51) is satisfied. Thus, (51) is redundant, and we are left with (53), which can be rewritten as the lower inequality in (46). This concludes the proof of the claim. □

Having proved the claim, the lemma follows. □

5. Main result and applications

We summarize the results of the previous sections as the main result of the paper.

Theorem 5.1. *Let g, r, d, r', d' be integers satisfying*

$$g \geq 3, \;\; r \geq 1, \;\; r' \geq 1, \;\; 2 \leq d \leq g-1, \;\; 2 \leq d' \leq g-1, \tag{54}$$

$$d \geq \begin{cases} \frac{g-3}{2} + 2r, & \textit{if } r \geq 2, \\ \frac{g}{2}, & \textit{if } r = 1, \end{cases} \tag{55}$$

$$(r-1)(3g-4)^2 < 2d^2(g-2), \tag{56}$$

$$\rho(g, r', d') < 0, \tag{57}$$

$$\begin{cases} d - r + r' - d' > \frac{r}{r'} - 1, & \textit{if } r' \leq r, \\ g + r' - 1 - d' - \frac{g+r-d-1}{r'-r+1} > 0, & \textit{if } r < r'. \end{cases} \tag{58}$$

Then for any K3 surface $(S,H) \in \mathcal{K}^r_{g,d}$ with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$*, we have that $|H|$ contains smooth irreducible curves, and for any such $C \in |H|$ we have:*

(i) C has genus g,
(ii) $|\mathscr{O}_C(L)|$ is a base point free complete g^r_d, which is very ample if $r \geq 3$,
(iii) C carries no $g^{r'}_{d'}$.

Proof. Condition (56) implies (5), which allows us to construct a K3 surface $(S,H) \in \mathcal{K}^r_{g,d}$ with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$ as in §3. Conditions (54)-(56) are the conditions in Proposition 3.4, which guarantee that (S,H) satisfies decomposition rigidity. For $r \geq 2$, inequality (55) can be rewritten as $g+4r-2d \leq 3$, so that also the assumption that $g+4r-2d \neq 4$ in Theorem 2.5 is satisfied. Thus, by the latter, properties (i) and (ii) in the theorem are fulfilled. If (iii) is not fulfilled, then, due to (57), there would be integers r_1, r_2, d_1, d_2 satisfying (12)-(15). But condition (58) is (46) of Lemma 4.1, which then tells us that such integers cannot exist. Thus, (iii) is fulfilled. □

Remark 5.2. Remark 3.7 tells us that condition (56) can be replaced by the simplified assumption

$$g \geq \min\{r(r+1),\ 10(r-1)\} = \begin{cases} r(r+1), & \text{if } 2 \leq r \leq 7, \\ 10(r-1), & \text{if } r \geq 8. \end{cases} \tag{59}$$

We will now deduce Theorem 1 in the introduction from Theorem 5.1. To this aim, we first need two lemmas.

Lemma 5.3. *Assume that the triple (g,r,d) is associated to an expected maximal Brill–Noether locus, $g \geq 3$ and $(g,r,d) \neq (6,2,5)$. Then conditions* (55)-(56) *in Theorem 5.1 are satisfied.*

Proof. If $r = 1$, then $d = \lceil \frac{g}{2} \rceil$ by (2), so (55) is satisfied. If $r \geq 2$, we have $g \geq r(r+1)$ by (3). Remark 5.2 yields that (56) is satisfied. Using $(g,r) \neq (6,2)$, one can check that the inequality

$$r - 1 + \left\lceil \frac{gr}{r+1} \right\rceil \geq \frac{g-3}{2} + 2r$$

holds, whence (2) implies that (55) is satisfied. Hence, (55)-(56) in Theorem 5.1 are satisfied. □

Lemma 5.4. *Assume that the triple (g,r,d) is associated to an expected maximal Brill–Noether locus, with $2 \leq d \leq g-1$, and (r',d') a pair of integers such that $r' \geq 1$, $2 \leq d' \leq g-1$, $\rho(g,r',d') < 0$, and either*

(a) *(g,r',d') is associated to an expected maximal Brill–Noether locus, $r' \neq r$, or*
(b) *$r' = r$ and $d' \leq d-1$, or*
(c) *$r' = r+1$ and $d' \leq d+1$.*

Then, except for the cases

$$(g,r,d,r',d') \neq (6,2,5,1,3), (7,2,6,1,4), (8,1,4,2,7), (9,2,7,1,5), \tag{60}$$

condition (58) *in Theorem 5.1 is satisfied.*

Proof. It is straightforward to check that (58) is satisfied under assumptions (b) and (c). We are therefore left with case (a), where $r' \neq r$ and (g,r',d') is associated to an expected maximal Brill–Noether locus.

We first prove the lower inequality in (58). After substituting expressions for d and d' given by (2) the inequality is equivalent to

$$\left\lceil \frac{gr'}{r'+1} \right\rceil (r'+1-r) < \left\lceil \frac{gr}{r+1} \right\rceil + g(r'-r). \tag{61}$$

The latter is implied by

$$\left(\frac{gr'}{r'+1} + 1 \right) (r'+1-r) \leq \frac{gr}{r+1} + g(r'-r),$$

which is equivalent to

$$r' + 1 - r \leq \frac{gr(r'-r)}{(r+1)(r'+1)}. \tag{62}$$

Since $g \geq r'(r'+1) \geq (r+1)(r'+1)$ by (3), the latter is implied by

$$r' + 1 - r \leq r(r'-r),$$

which is easily seen to hold as soon as $r \geq 2$. This leaves the case $r = 1$, where (62) reads

$$g \geq \frac{2r'(r'+1)}{r'-1}.$$

Using the fact that $g \geq r'(r'+1)$ by (3), we see that the latter is satisfied as long as $r' \geq 3$. Recalling that $r' \geq 2$ (as $r' > r$), this leaves the case $(r, r') = (1, 2)$, in which case (61) reads

$$2\left\lceil \frac{2g}{3} \right\rceil < g + \left\lceil \frac{g}{2} \right\rceil. \tag{63}$$

One easily checks that this is satisfied, since we are assuming $(g, r, d, r', d') \neq (8, 1, 4, 2, 7)$. This finishes the proof of the lower inequality in (58).

We finally prove the upper inequality in (58). After substituting the expressions for d and d' given by (2), the inequality is equivalent to

$$\left\lceil \frac{r'g}{r'+1} \right\rceil + \frac{r}{r'} < \left\lceil \frac{rg}{r+1} \right\rceil + 1. \tag{64}$$

The latter inequality is implied by

$$\frac{r'g}{r'+1} + \frac{r}{r'} \leq \frac{rg}{r+1},$$

which can be rewritten as

$$\frac{r}{r'} \leq \frac{g(r-r')}{(r'+1)(r+1)}.$$

By (3) the latter inequality is implied by

$$\frac{1}{r'} \leq \frac{r-r'}{r'+1},$$

which is equivalent to

$$\frac{r'+1}{r'} \leq r - r'.$$

Since $r' < r$, the latter is satisfied unless $r' = r - 1$. We have therefore left to prove (64) when $r' = r - 1$, in which case it reads

$$\left\lceil \frac{(r-1)g}{r} \right\rceil + \frac{1}{r-1} < \left\lceil \frac{rg}{r+1} \right\rceil. \tag{65}$$

Recall that $r \geq 2$ (as $r \geq r' + 1$). If $r \geq 3$, then (65) is implied by

$$\frac{(r-1)g}{r} + 1 \leq \frac{rg}{r+1},$$

which can be rewritten as $g \geq r^2 + r$, which holds by (3). Thus, (65) is proved if $r \geq 3$. If $r = 2$, then (65) reads

$$\left\lceil \frac{g}{2} \right\rceil + 1 < \left\lceil \frac{2g}{3} \right\rceil.$$

Using our assumptions that $(g, r, d, r', d') \neq (6, 2, 5, 1, 3), (7, 2, 6, 1, 4), (9, 2, 7, 1, 5)$, one verifies that the latter inequality is satisfied. Thus, (65) is proved if $r = 2$. This finishes the proof of the upper inequality in (58). □

Remark 5.5. While the hypotheses of Theorem 5.1 do not hold in the case $(g, r, d) = (6, 2, 5)$, we still have the existence of a polarized K3 surface $(S, H) \in \mathcal{K}^2_{6,5}$ such that $|H|$ contains smooth irreducible curves of genus 6 and $|\mathscr{O}_C(L)|$ is a base point free very ample complete g^2_5. Indeed, the existence of (S, H) with $\mathrm{Pic}(S) = \Lambda^2_{6,5}$ such that the conclusions (i)–(ii) of Theorem 5.1 are satisfied follows as above; this has, in fact, already been done (in a more general setting) in [16, Thm. 4.3]. In particular, the curves are isomorphic to smooth plane quintics, whence do not carry g^1_3s, so they again do not carry any $g^{r'}_{d'}$ with $(r', d') \neq (2, 5)$. One may also check that (S, H) satisfies decomposition rigidity.

Finally, applying all this to the expected maximal Brill–Noether loci, we can now give a proof of Theorem 1.

Proof of Theorem 1. Lemma 5.3 tells us that (g, r, d) satisfies conditions (54)-(56) in Theorem 5.1. In particular, by Proposition 3.4, the hypotheses in Theorem 2.5 are satisfied by any K3 surface S with $\mathrm{Pic}(S) = \Lambda^r_{g,d}$, as in §1.2. Let $C \in |H|$ be any smooth curve and assume it carries a $g^{r'}_{d'}$ with $\rho(g, r', d') < 0$ and $(r', d') \neq (r, d)$, hence C also carries a $g^{r''}_{d''}$ with (g, r'', d'') associated to an expected maximal Brill–Noether locus, obtained using the trivial containments $\mathcal{M}^r_{g,d} \subset \mathcal{M}^r_{g,d+1}$ and $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r-1}_{g,d-1}$, along with Serre duality.

Assume that $(r'', d'') \neq (r, d)$. Then, if $(g, r, d) \neq (6, 2, 5)$, Lemma 5.4 tells us that all the remaining conditions in Theorem 5.1 are satisfied for $(r', d') = (r'', d'')$ (recall that we are assuming $(g, r, d) \neq (7, 2, 6), (8, 1, 4), (9, 2, 7)$). Hence, by the same theorem, we get the desired contradiction, that is, that the $g^{r''}_{d''}$ cannot exist.

Assume that $(r'', d'') = (r, d)$. Then the g^r_d is obtained from the $g^{r'}_{d'}$ by a series of trivial containments and Serre duality, the last of which is Serre duality or one of the trivial containments $\mathcal{M}^r_{g,d-1} \subset \mathcal{M}^r_{g,d}$ or $\mathcal{M}^{r+1}_{g,d+1} \subset \mathcal{M}^r_{g,d}$. However, if Serre duality were the last step, then the $g^{g-d+r-1}_{2g-2-d}$ was obtained using trivial containments from the Serre duals of a g^{r+1}_{d+1} or a g^r_{d-1}. Thus, in any case, C carries a g^r_{d-1} or a g^{r+1}_{d+1}. Again if $(g, r, d) \neq (6, 2, 5)$, Lemma 5.4 tells us that all the remaining conditions in Theorem 5.1 are satisfied for $(r', d') = (r, d - 1)$ or $(r + 1, d + 1)$, and we get a contradiction again.

The remaining case of $(g, r, d) = (6, 2, 5)$ has already been handled by a direct geometric argument in [2, Proposition 6.1]. One can, in fact, also handle this case using K3 surfaces, as in Remark 5.5. □

Remark 5.6. Theorem 5.1 directly shows that the general curve in the Brill–Noether loci $\mathcal{M}^1_{7,4}$, $\mathcal{M}^2_{8,7}$, and $\mathcal{M}^1_{9,5}$ is g^r_d-general for $(r, d) = (1, 4), (2, 7), (1, 5)$, respectively, which are the maximal Brill–Noether loci in the exceptional genera, as in Example 1.3. Hence Theorem 1 says that any expected maximal Brill–Noether locus $\mathcal{M}^r_{g,d}$ contains a g^r_d-general curve, unless $g = 7, 8, 9$.

Remark 5.7. This also answers the question of containments of Brill–Noether loci of the form $\mathcal{M}^r_{g,d} \subset \mathcal{M}^{r'}_{g,d'}$ with $\rho(g, r, d) = -2$ and $\rho(g, r', d') = -1$, studied in [3, 9, 44]. Indeed, as observed in [7, Lemma 1.2], Brill–Noether loci with $\rho = -1, -2$ are expected maximal.

We give a few examples in genus 7, 8, 9 of distinguishing Brill–Noether loci using Theorem 5.1.

Example 5.8. In genus 7, Theorem 5.1 yields that the locus $\mathcal{M}^2_{7,6}$ is not contained in any Brill–Noether locus except $\mathcal{M}^1_{7,4}$, which is the unexpected containment of Example 1.3.

In genus 8, we obtain the non-containment $\mathcal{M}^1_{8,4} \not\subseteq \mathcal{M}^2_{8,6}$, thus the locus $\mathcal{M}^1_{8,4}$ is not contained in any Brill–Noether locus except $\mathcal{M}^2_{8,7}$, again an unexpected containment of Example 1.3. The containment $\mathcal{M}^2_{8,6} \subset \mathcal{M}^1_{8,4}$ is obtained in [34, Lemma 3.4].

In genus 9, we obtain that the locus $\mathcal{M}^2_{9,7}$ is not contained in any Brill–Noether locus except $\mathcal{M}^1_{9,5}$.

Theorem 5.1 also gives non-containments of non-maximal Brill–Noether loci.

Example 5.9. In genus 11, we obtain the non-containments $\mathcal{M}^3_{11,10} \nsubseteq \mathcal{M}^2_{11,8}$ and $\mathcal{M}^2_{11,8} \nsubseteq \mathcal{M}^3_{11,10}$.

In fact, Theorem 5.1 gives infinitely many non-containments of Brill–Noether loci that are not maximal. As a sample, let us see what happens when we lower the maximal degrees a little. Recall the definition of $d_{max}(g, r)$ from (2).

Proposition 5.10. *Let $\mathcal{M}^r_{g,d_{max}(g,r)}$ and $\mathcal{M}^{r'}_{g,d_{max}(g,r')}$ be expected maximal loci with $r \geq 2$, $r' \geq 2$ and $r \neq r'$. We have a non-containment $\mathcal{M}^r_{g,d} \nsubseteq \mathcal{M}^{r'}_{g,d_{max}(g,r')-1}$ in any of the following cases:*

- *$d \geq d_{max}(g, r) - 1$ and*
  - *$r' < r$, $(r, r', g) \neq (3, 2, 12)$, or*
  - *$(r, r') = (2, 3)$, $g \in \{13, 14, 15, 16, 18\}$,*
- *$d \geq d_{max}(g, r) - 2$ and $(r, r') = (2, 4)$, $g \in \{20, 21, 22, 24\}$,*
- *$d \geq d_{max}(g, r) - 3$ and $(r, r') = (2, 5)$, $g = 30$,*
- *$d \geq d_{max}(g, r) - (r' - r + 1)$ and*
  - *$r' > r \geq 3$, or*
  - *$r' \geq 6$, $r = 2$, or*
  - *$(r, r') = (2, 5)$ and $g \geq 31$, or*
  - *$(r, r') = (2, 4)$ and $g \geq 25$, or $g = 23$, or*
  - *$(r, r') = (2, 3)$ and $g \geq 19$, or $g = 17$.*

Proof. We have proved that(58) holds for $d = d_{max}(g, r)$ and for $d' = d_{max}(g, r')$. Setting instead $d' = d_{max}(g, r') - 1$, we therefore see that the upper inequality in (58) holds for $d \geq d_{max}(g, r') - 1$, whereas the lower one hold for $d \geq d_{max}(g, r) - (r' - r + 1)$. Since $g \geq r(r + 1)$ by maximality of $\mathcal{M}^r_{g,d_{max}(g,r)}$ (cf. (3)), Remark 5.2 tells us that (56) is redundant. Hence, it remains to check (55) by substituting for the values of d and using $g \geq r(r + 1)$ and $g \geq r'(r' + 1)$. We leave the details to the reader. □

Example 5.11. In genus 14, Proposition 5.10 gives the non-containments $\mathcal{M}^3_{14,12} \nsubseteq \mathcal{M}^2_{14,10}$ and $\mathcal{M}^2_{14,10} \nsubseteq \mathcal{M}^3_{14,12}$.

We also give an example where Theorem 5.1 does not suffice to prove a non-containment.

Example 5.12. Let $(g, r, d) = (9, 2, 6)$ and $(g, r', d') = (9, 3, 8)$. We cannot apply Theorem 2.5, as on the lattice $\Lambda^2_{9,6}$, we have $(H - 2L)^2 = 0$, hence (S, H) does not satisfy decomposition rigidity. However, from [3, Proposition 2.5], we obtain $\mathcal{M}^2_{9,6} \nsubseteq \mathcal{M}^3_{9,8}$.

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