ZETA FUNCTIONS OF K3 CATEGORIES OVER FINITE FIELDS

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Abstract. We study the arithmetic of the K3 category associated to a cubic fourfold over a non-algebraically closed field k, specifically, the Galois representation on its ℓ-adic Mukai realization. For k a finite field, we define the zeta function of a general K3 category, an invariant under Fourier–Mukai equivalence that can be used to study its geometricity. We show both how the zeta function can obstruct the geometricity of a K3 category, as well as fail to detect nongeometricity. Finally, we study an analogue of Honda–Tate for K3 surfaces and for K3 categories, and provide a nontrivial restriction on the possible Weil polynomials of the K3 category of a cubic fourfold.

Introduction

Given a smooth cubic fourfold $X \subset \mathbb{P}^5$ over a field k, Kuznetsov [39] has established a semiorthogonal decomposition of its derived category

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{A}_X, \mathscr{O}, \mathscr{O}(1), \mathscr{O}(2)\rangle.$$

The admissible subcategory A_X is known as the K3 category of the cubic fourfold, cf. [31], in light of Kuznetsov's proof that A_X is a Calabi–Yau category of dimension 2 in the sense of Kontsevich [38] and has the same Hochschild cohomology as a K3 surface. Such categories are called noncommutative K3 surfaces, see [44, Section 2.2]. We say that A_X is geometric over k if there is a k-linear equivalence between A_X and $\mathsf{D}^{\mathrm{b}}(S)$ for a K3 surface S defined over k. When $k = \mathbb{C}$, Kuznetsov conjectured that X is rational if and only if A_X is geometric. As evidence for his conjecture, Kuznetsov checked it for the known families of rational cubic fourfolds, and more recent work, e.g., [3], [4], [6], [10], has established additional cases and shown the conjecture to be equivalent to a Hodge-theoretic characterization of rationality for cubic fourfolds advocated by Hassett and Harris.

In this note, we introduce point counting on X as a tool to study the geometricity of the K3 category A_X. Specifically, for X defined over a finite field k, we define the notion of point count $|\mathsf{A}_X(k)|$ of the K3-category A_X, which is an integer that is a derived invariant of A_X under Fourier–Mukai equivalence.

In the geometric case, when $\mathsf{A}_X \cong \mathsf{D}^{\mathrm{b}}(S)$ for a K3 surface S defined over k, we recover the classical point count $|\mathsf{A}_X(k)| = |S(k)|$ of S. But in general, the point count of A_X may be negative or fail to satisfy other necessary growth conditions on point counts of varieties.

More generally, we define point counts and zeta functions for any noncommutative surface, see Section 1. Our main motivation is that point counts can obstruct geometricity of A_X. Indeed, when X is a cubic fourfold over $\mathbb{Q}$ with good reduction at p, such that $\mathsf{A}_{X_{\mathbb{F}_p}}$ is not geometric over $\mathbb{F}_p$, then A_X is not geometric over $\mathbb{Z}_p$, i.e., there is no $\mathbb{Z}_p$-linear equivalence $\mathsf{A}_{\mathcal{X}} \cong \mathsf{D}^{\mathrm{b}}(\mathcal{S})$ for smooth proper models $\mathcal{X}$ of X and $\mathcal{S}$ of S over $\mathbb{Z}_p$. In this setting, it is expected that any Fourier–Mukai equivalence $\mathsf{A}_X \cong \mathsf{D}^{\mathrm{b}}(S)$ over $\mathbb{Q}$ spreads to a Fourier–Mukai equivalence over $\mathbb{Z}_p$ as long as S has good reduction at p, see [47, Desideratum 3.4.4]. This would imply that whenever $\mathsf{A}_{X_{\mathbb{F}_p}}$ is not geometric over $\mathbb{F}_p$, e.g., has negative point counts, then X does not admit any associated K3 surface over $\mathbb{Q}$ with good reduction at p.

Using the census of cubic fourfolds over $\mathbb{F}_2$ obtained in [7], we can give a lower bound on the number of isomorphism classes of cubic fourfolds X over $\mathbb{F}_2$ whose K3 category is not geometric, either because they have negative point counts (i.e., $|\mathsf{A}_X(\mathbb{F}_{2^n})| < 0$ for some $n \geq 1$) or because they fail field extension growth conditions (i.e., $|\mathsf{A}_X(\mathbb{F}_{2^{mn}})| < |\mathsf{A}_X(\mathbb{F}_{2^n})|$ for some $m, n \geq 1$).

Theorem 1. *Of the* $1\,069\,562$ *isomorphism classes of smooth cubic fourfolds over* $\mathbb{F}_2$*, we find that:*

(a) 2662 *have K3 categories with negative point counts; of these,* 436 *are ordinary and exactly one is Noether–Lefschetz general.*
(b) 2343 *have K3 categories with nonnegative point counts but fail the field extension growth condition; of these,* 1084 *are ordinary and* 140 *are Noether–Lefschetz general.*

In particular, only 0.47% *of smooth cubic fourfolds over* $\mathbb{F}_2$ *have nongeometric K3 category because they have either negative point counts or fail field extension growth.*

Addington informed us that he already observed the existence of K3 categories with negative point counts in the course of computer experiments for [1].

On the other hand, we expect that point counting alone can sometimes fail to obstruct geometricity. As evidence, we show the following (see Theorem 4.1).

Theorem 2. *There exist special cubic fourfolds* X *over* $\mathbb{Q}$ *such that:*

- X *has good reduction at* 2 *and* $\mathsf{A}_{X_{\mathbb{F}_2}}$ *has all positive point counts with field extension growth conditions, and*
- $\mathsf{A}_{X_{\mathbb{C}}}$ *is not equivalent to* $\mathsf{D}^b(S, \alpha)$ *for any K3 surface* S *defined over* $\mathbb{C}$ *and any Brauer class* $\alpha \in \mathrm{Br}(S)$*.*

In addition, about 99.87% of Noether–Lefschetz general cubic fourfold over $\mathbb{F}_2$ satisfy the conditions in Theorem 2 (see [31, Theorem 1.4]). The existence of special such cubic fourfolds indicate that a potential Honda–Tate theory for K3 surfaces (see Section 3) is still quite mysterious, as our current necessary conditions (e.g., in [33]) on the zeta function of a K3 surface hold for such potentially noncommutative examples.

Our notion of point count of a K3 category may have additional applications in the formulation of a Honda-Tate theory for noncommutative K3 surfaces. As a demonstration of the potential applications, we first note that if A_X is the K3 category of a cubic fourfold X, then the categorical Hilbert square $\mathsf{A}_X^{[2]}$ has point-counts arising from geometry: there is an equality of zeta functions of $\mathsf{A}_X^{[2]}$ and of the Fano variety $F(X)$ of lines of a cubic fourfold X over a finite field. We provide a proof of this (see Proposition 2.9), though it is also deducible from other work (see Remark 2.10). The geometricity of the Hilbert square gives a nontrivial condition on whether a zeta function can come from the K3 category of a cubic fourfold. Indeed, of the $2\,971\,182$ polynomials in [33, Computation 3a] that are potentially the Weil polynomial of a noncommutative K3 surface over $\mathbb{F}_2$, there are $31\,256$ that cannot arise from a cubic fourfold because of failure of field-extension growth of their Hilbert square point counts. See Section 3 for a more detailed discussion of this Honda-Tate theory.

Several authors have considered derived categories and point counts over finite fields. In [50, Conjecture 1], Orlov conjectured that derived equivalent varieties have the same effective Chow motive, and hence should have the same point counts. For any variety with ample or anti-ample canonical class, derived equivalence implies isomorphism [13], hence the zeta function is a derived invariant. Several other cases of derived invariance of point counts have been established: by Antieau, Krashen, and Ward [5] for curves of genus 1, by Lieblich and Olsson [43] for K3 surfaces, and by Honigs [28, 29] for all abelian varieties and all smooth projective surfaces and threefolds. This derived invariance of point counts is especially surprising for threefolds, since the Hodge numbers of a threefold are not a derived invariant in positive characteristic [2].

The key to proving the derived invariance of point counts in these known cases is to consider the action of Frobenius on the ℓ-adic Mukai realization $\bigoplus_i H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(\lceil \frac{i}{2} \rceil))$, a Galois module that only depends on the derived equivalence class of X. More specifically, let k be a finite field with q elements, let X be a smooth projective k-variety, and let $\overline{X} = X_{\overline{k}}$. Let $F\colon \overline{X} \to \overline{X}$ denote the relative q-power Frobenius. Since X is smooth and projective, the action of F^* on étale cohomology satisfies the Grothendieck–Lefschetz trace formula

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{2\dim X} (-1)^i \mathrm{tr}(F^* \mid H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)).$$

The cohomology of low dimensional varieties is simple enough that one can extract, from the action of F^* on the ℓ-adic Mukai realization, just enough information about the eigenvalues of Frobenius on the individual H^i to conclude derived invariance of point counts. Our definition of the point counts of a K3 category is inspired by this idea.

In this note, we first introduce the notion of point counts of a Calabi–Yau category of dimension 2 in Section 1. We prove some basic properties, including Fourier–Mukai invariance. Then, in Section 2, we apply our notion to the study of cubic fourfolds and their associated noncommutative K3s. We give examples of noncommutative K3s defined over $\mathbb{F}_2$ with negative point counts, which is an obstruction to geometricity of the K3 category. We also consider the relationship between the Fano variety of lines on the cubic and the Hilbert square of its K3 category, finding that these point counts will always agree. Finally, in Section 4, we give an example of a nonadmissible special cubic fourfold over $\mathbb{Q}$ whose associated noncommutative K3, when reduced modulo 2, has the zeta function of a usual K3 surface. Our example illustrates that the zeta function is too coarse as invariant to distinguish admissible and nonadmissible cubics.

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1. Galois modules from admissible subcategories

Let X be a smooth projective variety defined over a perfect field k and ℓ a prime not equal to the characteristic of k. Let $\iota_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$ denote the embedding of some admissible subcategory of the bounded derived category of X, and denote by $\pi_{\mathsf{C}}\colon \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{C}$ the left adjoint of ι_{C}. Suppose further that this functor ι_{C} is

k-linear. From the derived category of X, one can recover the even and odd Mukai structures:

$$\widetilde{H}^{\mathrm{even}}(X) := \bigoplus_i H^{2i}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(i)),\ \ \widetilde{H}^{\mathrm{odd}}(X) := \bigoplus_i H^{2i-1}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(i))$$

and the total Mukai structure

$$\widetilde{H}(X) := \widetilde{H}^{\mathrm{even}}(X) \oplus \widetilde{H}^{\mathrm{odd}}(X).$$

Definition 1.1. We define the ℓ-adic Galois module of a k-linear admissible embedding of a subcategory $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$ to be the G_k-submodule of $\widetilde{H}(X)$ which is the image of the cohomological Fourier–Mukai transform $(\iota_{\mathsf{C}} \circ \pi_{\mathsf{C}})^H : \widetilde{H}(X) \to \widetilde{H}(X)$. We denote this submodule by $\widetilde{H}(\mathsf{C})$.

Remark 1.2. We have defined $\widetilde{H}(C)$ as a $\mathbb{Q}_\ell$-vector space; we do not concern ourselves with defining an underlying integral structure in the present work.

Remark 1.3. Note that the above definition depends *a priori* on the embedding $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$. We will often fix an embedding $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$ and refer to $\widetilde{H}(\mathsf{C})$ as simply the Galois module associated to C, with the embedding implicitly understood. However, we will soon see that $\widetilde{H}(\mathsf{C})$ only depends on the embedding $\mathsf{C} \to \mathsf{D}^{\mathrm{b}}(X)$ up to Fourier–Mukai equivalence of C. The ℓ-adic realization functor on noncommutative motives, see e.g., [12, Section 3.7], should provide an alternative way of constructing $\widetilde{H}(\mathsf{C})$ independent of the embedding.

Definition 1.4. Given a $\mathbb{Q}_\ell$ vector space V with an action of a linear operator φ, we define the associated L-polynomial by

$$L(T; \varphi) := \det(\mathrm{Id} - \varphi T).$$

Given an admissible embedding $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$, we define the L-polynomial

$$L_{\widetilde{H}(\mathsf{C})}(T) := L(T; F^*).$$

associated to the action of F^* on $\widetilde{H}(\mathsf{C})$.

1.1. **Point counts on a noncommutative K3 surface.** If one views C as a noncommutative variety, is there a meaningful definition for $\mathsf{C}(\mathbb{F}_q)$ or $|\mathsf{C}(\mathbb{F}_q)|$? Unfortunately, there is not in general an obvious way to compute Lefschetz traces from $\widetilde{H}(\mathsf{C})$, because the cohomological Fourier–Mukai transform in general does not preserve cohomological degree. We still manage to define a notion of "point count" below, and this notion will agree with classical point counts when C is the bounded derived category of a K3 surface.

For the rest of this section, let k be a field with q elements and X a smooth projective variety over k. Let $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$ be an admissible subcategory defined over a finite field k. Assume C is a Calabi–Yau category of dimension 2 (so the shift [2] is a Serre functor) with the same Hochschild cohomology as the bounded derived category of a K3 surface, what we often call a noncommutative K3 surface, see [44, Section 2.2].

Definition 1.5. The zeta function of a noncommutative K3 surface C, denoted $Z_{\mathsf{C}}(T)$, is the rational function

$$Z_{\mathsf{C}}(T) := \frac{(1-qT)^2}{(1-T) \cdot L_{\widetilde{H}(\mathsf{C})}(qT) \cdot (1-q^2T)}.$$

For $K \supseteq k$ a field extension of degree n, define the K-point count of C by

$$|\mathsf{C}(K)| := na_n,$$

where a_n is the coefficient of T^n in the formal series expansion of $\log(Z_{\mathsf{C}}(T))$.

These point counts are rational numbers, but are not necessarily integral or positive.

Remark 1.6. If we further assume that C is an ℓ-adic noncommutative K3 surface, i.e., $\widetilde{H}(\mathsf{C})$ admits a decomposition $H^0 \oplus H^2 \oplus H^4$, where $H^0 = H^4 = \mathbb{Q}_\ell$ have trivial Galois action, then $L_{\widetilde{H}(\mathsf{C})}(T)$ has two factors of $(1-T)$ corresponding to H^0 and H^4, and $L_{\mathsf{C}}(T) := L_{\widetilde{H}(\mathsf{C})}/(1-T)^2$ is a polynomial that corresponds to the H^2. This is the case for the K3 category A_X of a cubic fourfold X since A_X is the semiorthogonal complement of an exceptional collection, and we wonder whether it holds for every noncommutative K3 surface.

1.2. **FM-invariance.** By a Fourier–Mukai equivalence between admissible subcategories $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$ and $\mathsf{C}' \hookrightarrow \mathsf{D}^{\mathrm{b}}(X')$, we mean an exact equivalence $\mathsf{C} \cong \mathsf{C}'$ which appears in the factorization of a Fourier–Mukai transform between X and X':

$$\mathsf{D}^{\mathrm{b}}(X) \xrightarrow{\pi_C} \mathsf{C} \xrightarrow{\sim} \mathsf{C}' \xrightarrow{\iota_{C'}} \mathsf{D}^{\mathrm{b}}(X').$$

Our definition of point count should be invariant under such equivalences. This would be automatic if $\mathsf{C} \to \mathsf{C}'$ were induced from an equivalence $\mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(X')$, but in general there can be equivalences of admissible subcategories that don't arise from an equivalence of their ambient derived categories. Still, we can show that *Fourier–Mukai* equivalences induce isomorphisms $\widetilde{H}(\mathsf{C}_X) \to \widetilde{H}(\mathsf{C}_{X'})$.

Proposition 1.7. *If $\mathsf{C} \to \mathsf{C}'$ is a k-linear Fourier–Mukai equivalence, induced by some kernel $\mathcal{E} \in \mathsf{D}^b(X \times X')$ defined over k, then $\widetilde{H}(\mathsf{C}) \to \widetilde{H}(\mathsf{C}')$ is an isomorphism of Galois modules.*

Proof. The FM-equivalence $F\colon \mathsf{C} \to \mathsf{C}'$ fits into the sequence of exact functors

$$\mathsf{D}^{\mathrm{b}}(X) \xrightarrow{\pi_X} \mathsf{C} \xrightarrow{F} \mathsf{C}' \xrightarrow{\iota_{X'}} \mathsf{D}^{\mathrm{b}}(X')$$

whose overall composition is Fourier–Mukai. Let $\mathcal{E} \in \mathsf{D}^{\mathrm{b}}(X \times X')$ be the kernel of the composition, inducing the exact functor $\Phi_E\colon \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(X')$, which by hypothesis is defined over k. Then the right adjoint functor $\Phi_{\mathcal{E}_R}\colon \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(X')$ induces an inverse k-linear equivalence $\mathsf{C}' \to \mathsf{C}$. The Fourier–Mukai kernels induce maps $\Phi^H_{\mathcal{E}}\colon \widetilde{H}(X) \to \widetilde{H}(X')$ and $\Phi^H_{\mathcal{E}_R}\colon \widetilde{H}(X') \to \widetilde{H}(X)$; both of these kernels are defined over k and are therefore compatible with the action of Frobenius, in the sense that $F^*\mathcal{E} = \mathcal{E}$. Since $\Phi_{\mathcal{E}} \circ \Phi_{\mathcal{E}_R}$ restricted to C is naturally isomorphic to the identity on C, then we have that the cohomological transform $\Phi^H_{\mathcal{E}} \circ \Phi^H_{\mathcal{E}_R}\colon \widetilde{H}(X) \to \widetilde{H}(X)$ is compatible with Froebnius and acts as the identity on $\widetilde{H}(\mathsf{C})$, and thus $\Phi^H_{\mathcal{E}}$ restricted to $\widetilde{H}(\mathsf{C})$ yields an isomorphism of Galois modules $\widetilde{H}(\mathsf{C}) \to \widetilde{H}(\mathsf{C}')$. $\square$

Since our definition of $|\mathsf{C}(\mathbb{F}_q)|$ depends only on the Galois module $\widetilde{H}(\mathsf{C})$, we conclude the FM-invariance of point counts.

Corollary 1.8. *Let k be a finite field. If there is k-linear exact equivalence $\mathsf{C} \to \mathsf{C}'$ which is Fourier–Mukai, then $Z_{\mathsf{C}} = Z_{\mathsf{C}'}$ and $|\mathsf{C}(K)| = |\mathsf{C}'(K)|$ for any finite extension $K \supseteq k$.*

1.3. **Geometricity of K3 categories.** Point-counting can help detect whether C is derived equivalent to a (twisted) K3 surface over k.

Corollary 1.9. *Let k be a finite field. If $\mathsf{C} \cong \mathsf{D}^b(S,\alpha)$ is a k-linear FM equivalence for some twisted K3 surface (S,α) defined over k, then $Z_{\mathsf{C}} = Z_S$; in particular, $|\mathsf{C}(K)| = |S(K)|$ for any finite extension $K \supseteq k$.*

Proof. As in the proof of Proposition 1.7, there is an isomorphism of Galois modules $\widetilde{H}(\mathsf{C}) \cong \widetilde{H}(S,\alpha)$, and since these are $\mathbb{Q}_\ell$-vector spaces we further have an isomorphism of Galois modules $\widetilde{H}(S,\alpha) = \widetilde{H}(S)$, from which one recovers the point counts of S [28]. □

The main application of the above corollary is to obstruct geometricity of the Calabi–Yau category C: if the point counts are nonintegral or negative, then C cannot be derived equivalent over k to a (twisted) K3 surface.

2. The K3 category of a cubic fourfold

In the case where X is a cubic fourfold defined over a finite field k with q elements, and $\mathsf{A}_X \hookrightarrow \mathsf{D}^{\mathrm{b}}(X)$ is the admissible embedding of its K3 category into its derived category, we wish to study the point counts of A_X.

Let $L_4(T) = \det(\mathrm{Id} - TF^* \mid H^4_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(2)))$ denote the L-polynomial on the middle cohomology of X, which is a polynomial of degree 23 whose roots have complex absolute value 1. Let $L_{4,pr}(T) = L_4(T)/(1-T)$ denote the factor corresponding to the primitive cohomology. Then we have

$$Z_{\mathsf{A}_X}(T) = \frac{1-qT}{(1-T)L_4(qT)(1-q^2T)} = \frac{1}{(1-T)L_{4,pr}(qT)(1-q^2T)}.$$

We remark that this gives the following formula for the point counts of A_X in terms of the point counts of X

$$|\mathsf{A}_X(\mathbb{F}_{q^n})| = \frac{|X(\mathbb{F}_{q^n})| - 1 - q^{2n} - q^{4n}}{q^n}$$

and inversely, a formula for the point counts of X in terms of those for A_X

$$|X(\mathbb{F}_{q^n})| = 1 + q^n|\mathsf{A}_X(\mathbb{F}_{q^n})| + q^{2n} + q^{4n} \tag{1}$$

These formulae, together with Chevalley–Warning–Ax theorem [9], which gives that $|X(\mathbb{F}_q)| \equiv 1 \pmod q$, imply the following.

Lemma 2.1. *Let $\mathsf{A}_X \hookrightarrow \mathsf{D}^b(X)$ be the K3 category of a cubic fourfold X over a finite field k, and let K/k be a finite extension. Then the point count $|\mathsf{A}_X(K)|$ is an integer.* □

Remark 2.2. Li, Pertusi, and Zhao [41] show, at least over the complex numbers, that any exact equivalence of K3 categories of cubic fourfolds is Fourier–Mukai. If one could show this over finite fields, then consequently the point counts of A_X would be independent of the admissible embedding. The missing ingredient over a finite field is the nonemptiness of a certain moduli space of objects in the K3 category, but this would take us too far afield from our primary focus.

2.1. **Negative point counts and obstructions to geometricity.** In the database [8], we have examples of cubic fourfolds over $\mathbb{F}_2$ whose K3 categories have some negative point counts $|\mathsf{A}_X(\mathbb{F}_2)| < 0$. This could be indicative of some associated K3 defined over some larger base extension, or it could indicate that there is no associated K3 even geometrically. In any case, we can definitively say that these cubics have no associated K3 defined over $\mathbb{F}_2$.

Computation 2.3. *There are* 2662 *cubic fourfolds up to isomorphism over* $\mathbb{F}_2$ *for which* $|\mathsf{A}_X(K)| < 0$ *for some finite extension of* K*, and hence do not have associated K3 over the field* $\mathbb{F}_2$.

In fact, all but one of these cubics is Noether–Lefschetz special, demonstrating that negative point counts can be used to exhibit explicit special cubic fourfolds with no associated K3 over $\mathbb{F}_2$.

Computation 2.4. *There is a unique* $\mathbb{F}_2$*-isomorphism class of smooth cubic fourfold* X *defined over* $\mathbb{F}_2$ *that is geometrically Noether–Lefschetz general and whose K3 category* A_X *has negative point counts, represented by*

$$x_1^2x_2 + x_1^2x_6 + x_1x_2x_6 + x_1x_3x_5 + x_1x_4^2 + x_1x_5^2 + x_1x_6^2 + x_2^3 + x_2^2x_5 + x_2^2x_6$$
$$+x_2x_3x_4 + x_2x_5^2 + x_3^3 + x_3x_6^2 + x_4^2x_5 + x_4x_6^2 + x_6^3.$$

Remark 2.5. There are 436 among those from Computation 2.3 which are ordinary. The cubic fourfold of Computation 2.4 is a non-ordinary cubic fourfold (of height 7).

Remark 2.6. We also found that there are a handful of cubics for which the $\mathbb{F}_2$-point counts of A_X are positive but for which $|\mathsf{A}_X(\mathbb{F}_{2^m})| < 0$ for some $m > 1$.

Another natural condition on the point counts for a K3 surface S, considered by Kedlaya and Sutherland [33], is that $|S(\mathbb{F}_{q^{mn}})| \geq |S(\mathbb{F}_{q^n})|$ for all $m, n \geq 1$.

This provides, for a cubic fourfold X, an obstruction to geometricity over k.

Computation 2.7. *There are 2343 cubic fourfolds over* $\mathbb{F}_2$ *that have* $|\mathsf{A}_X(\mathbb{F}_{2^k})| > 0$ *for all* $k \geq 1$ *but* $|\mathsf{A}_X(\mathbb{F}_{2^{mn}})| < |\mathsf{A}_X(\mathbb{F}_{2^n})|$ *for some* $m, n \geq 1$*, and hence do not have associated K3 over* $\mathbb{F}_2$.

Remark 2.8. There are 1084 cubic fourfolds in the above computation which are ordinary.

2.2. **Hilbert schemes and Fano varieties.** For X a smooth cubic fourfold defined over the finite field $k = \mathbb{F}_q$ and for any finite extension $\mathbb{F}_{q^n} \supset k$, the point counts of X determine the point counts of its Fano variety of lines $F_1(X)$. Precisely, we have ([23, Corollary 5.2], [18, Equation 8]):

$$|F_1(X)(\mathbb{F}_{q^n})| = \frac{|X(\mathbb{F}_{q^n})|^2 - 2(1+q^{4n})|X(\mathbb{F}_{q^n})| + |X(\mathbb{F}_{q^{2n}})|}{2q^{2n}}. \tag{2}$$

On the other hand, if $S^{[2]}$ is the Hilbert scheme of length two subscheme on a K3 surface we have a formula for its point counts, which was first used by Göttsche in [25]:

$$|S^{[2]}(\mathbb{F}_{q^n})| = \binom{|S(\mathbb{F}_{q^n})|}{2} + (q^n+1)|S(\mathbb{F}_{q^n})| + \frac{|S(\mathbb{F}_{q^{2n}})| - |S(\mathbb{F}_{q^n})|}{2}$$

Ganter and Kapranov [24] defined the symmetric square $\mathsf{C}^{[2]}$ of any k-linear triangulated (or dg-)category C. When S is a smooth projective surface, one has a k-linear equivalence $\mathsf{D}^{\mathrm{b}}(S)^{[2]} \to \mathsf{D}^{\mathrm{b}}(S^{[2]})$, see [30, Chapter 7, Remark 3.28(i)]. This formally motivates the following definition of point counts of the Hilbert square $\mathsf{A}_X^{[2]}$ of the K3 category (or any noncommutative K3)

$$|\mathsf{A}_X^{[2]}(\mathbb{F}_{q^n})| := \binom{|\mathsf{A}_X(\mathbb{F}_{q^n})|}{2} + (q^n+1)|\mathsf{A}_X(\mathbb{F}_{q^n})| + \frac{|\mathsf{A}_X(\mathbb{F}_{q^{2n}})| - |\mathsf{A}_X(\mathbb{F}_{q^n})|}{2} \tag{3}$$

as well as the corresponding zeta function $Z_{\mathsf{A}_X^{[2]}}(T)$.

Proposition 2.9. *Let X be a cubic fourfold over a finite field k. Then there is a equality of zeta functions*

$$Z_{\mathsf{A}_X^{[2]}}(T) = Z_{F_1(X)}(T)$$

Proof. By the Weil conjectures, there are algebraic integers $\alpha_1, \ldots, \alpha_{22}$ such that

$$|X(\mathbb{F}_{q^n})| = 1 + q^n + \sum_{j=1}^{22} \alpha_j^n + q^{2n} + q^{3n} + q^{4n}, \tag{4}$$

$$|X(\mathbb{F}_{q^{2n}})| = 1 + q^{2n} + \sum_{j=1}^{22} \alpha_j^{2n} + q^{4n} + q^{6n} + q^{8n}.$$

For the K3 category, one has

$$|\mathsf{A}_X(\mathbb{F}_{q^n})| = 1 + \sum_{j=1}^{22} \left(\frac{\alpha_j}{q}\right)^n + q^{2n},$$

$$|\mathsf{A}_X(\mathbb{F}_{q^{2n}})| = 1 + \sum_{j=1}^{22} \left(\frac{\alpha_j}{q}\right)^{2n} + q^{4n}.$$

Evaluating these expressions into Equations 2 and 3, one find formally that

$$|\mathsf{A}_X^{[2]}(\mathbb{F}_{q^n})| = |F_1(X)(\mathbb{F}_{q^n})|$$

for all $n \geq 1$ and thereby giving the equality of zeta functions. □

Remark 2.10. The construction of the symmetric square implies that $\mathsf{A}_X^{[2]} \subset \mathsf{D}^{\mathrm{b}}(X^{[2]})$ is an admissible subcategory, where $X^{[2]}$ is the Hilbert scheme of length 2 subschemes on X, see [30, Chapter 7, Remark 3.28(ii)]. Work of Belmans, Fu, and Raedschelders [11, Theorem B] (cf. [32, Section 8]) presents both $\mathsf{A}_X^{[2]}$ and $\mathsf{D}^{\mathrm{b}}(F_1(X))$ as pieces of a semiorthogonal decomposition of $\mathsf{D}^{\mathrm{b}}(X^{[2]})$ with all other components equivalent to $\mathsf{D}^{\mathrm{b}}(X)$, from which one can also deduce the equality of L-polynomials $L_{\widetilde{H}(F_1(X))}(T) = L_{\widetilde{H}(\mathsf{A}_X^{[2]})}(T)$, thereby providing a different proof of Proposition 2.9.

It would be interesting to consider an appropriate Grothendieck ring of noncommutative varieties, which is a generalization of both the Grothendieck ring of k-varieties and the Grothendieck ring of differential graded k-linear categories, in which the formula (in analogy with work of Galkin and Shinder [23, Theorem 5.1], see [30, Chapter 7, Remark 3.28(ii)])

$$[X^{[2]}] = \mathbb{L}^2\,[\mathsf{A}_X^{[2]}] + [\mathbb{P}^4][X]$$

could reside, where $\mathbb{L} = [\mathbb{A}^1]$. One can check directly, in analogy with the proof of Proposition 2.9, that if k is a finite field, then

$$|\mathsf{A}_X^{[2]}(k)| = \frac{|X^{[2]}(k)| - |\mathbb{P}^4(k)||X(k)|}{q^2},$$

an equality that would follow from the existence of an appropriate point counting motivic measure on this Grothendieck ring. One could also ask whether the equality $[F_1(X)] = [\mathsf{A}_X^{[2]}]$ holds in this ring (and not just after inverting $\mathbb{L}$).

Finally, a k-equivalence $\mathsf{A}_X^{[2]} \cong \mathsf{D}^{\mathrm{b}}(F_1(X))$, which would immediately imply Proposition 2.9, is a conjecture attributed to Galkin (see [30, Chapter 7, Remark 3.28(iii)] and [32, Section 8]). Kemboi and Segal (in [36]) give such an equivalence over the complex numbers using matrix factorizations.

Remark 2.11. Frei [20] has already shown that point counts of moduli spaces of stable sheaves on K3 surfaces only depend on the dimension of the moduli space and the underlying K3. Li, Pertusi, and Zhao [40] have shown that $F_1(X)$ is a moduli space of Bridgeland stable objects in A_X. Therefore, our Proposition 2.9 is evidence that Frei's result should extend to the noncommutative setting: the point count of a moduli space of Bridgeland stable objects on a noncommutative K3 surface should only depend on the category and the dimension of the moduli space.

3. Towards a Honda–Tate for K3 surfaces

Kedlaya and Sutherland [33] have initiated a program that could be called Honda–Tate for K3 surfaces, in analogy with the classical Honda–Tate theorem [27, 53] for abelian varieties over a finite field. In this section, we give some further details of what this program entails and provide a few additional observations, including how varieties "of K3-type" (such as cubic fourfolds) might play a role.

For context, we will review the classical Honda–Tate theorem in the case of abelian varieties. For an abelian variety A of dimension g over $\mathbb{F}_q$, the zeta function $\zeta_A(T)$ is completely determined by, and determines, the characteristic polynomial $\Phi_A(T)$ of Frobenius on $H^1_{\text{ét}}(\overline{A}, \mathbb{Q}_\ell)$ for any ℓ prime to q. In this case, $\Phi_A(T)$ is a Weil polynomial of degree $2g$, all of whose roots have absolute value $q^{1/2}$ and must satisfy other arithmetic conditions. Isogenous abelian varieties have the same zeta function, and the map from the set of isogeny classes of abelian varieties of dimension g to the set of Weil polynomials is injective by a result of Tate [53]. The description of the image of the map, i.e., the set of Weil polynomials realized by abelian varieties of dimension g, is a result of Honda [27].

For a K3 surface S over $\mathbb{F}_q$, the zeta function is given by

$$\zeta_S(T) = \frac{1}{(1-T)L_S(qT)(1-q^2t)}$$

where $L_S(T)$ is the L-polynomial of Frobenius acting $H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell(1))$ for any ℓ prime to q. In this case, $L_S(T)$ is a Weil polynomial of degree 22, all of whose roots have absolute value 1 and must satisfy other constraints, see Theorem 3.2.

There is a classical notion of isogeny of K3 surfaces S and S' over $\mathbb{C}$, namely, an isometry $\varphi\colon H^2(S(\mathbb{C}), \mathbb{Q}) \to H^2(S'(\mathbb{C}), \mathbb{Q})$ of rational Hodge structures with intersection pairing. Various authors have proposed notions of isogeny between K3 surfaces over a finite field, e.g., [15] and [55], which would produce an isometry $\varphi\colon H^2_{\text{ét}}(\overline{S}, \mathbb{Q}_\ell) \to H^2_{\text{ét}}(\overline{S}', \mathbb{Q}_\ell)$ of Galois modules with intersection pairing for all ℓ prime to q. In particular, isogenous K3 surfaces should have the same Weil polynomial. Honda–Tate for K3 surfaces then consists of the following two problems.

Problem 3.1 (Honda–Tate for K3s)**.**

(1) "Tate for K3s" Determine whether the map from isogeny classes of K3 surfaces over $\mathbb{F}_q$ to Weil polynomials is injective.
(2) "Honda for K3s" Determine the Weil polynomials that arise from K3 surfaces over $\mathbb{F}_q$.

Tate for K3s is likely known to the experts and should follow from the semisimplicity of Frobenius acting on ℓ-adic cohomology, whereas Honda for K3s seems to be wide open, and is the subject of Kedlaya and Sutherland's computational work [33], as well as work by Taelman [52] and Ito [34]. In particular, Kedlaya and Sutherland describe an algorithm to generate a list of all polynomials that could potentially arise as Weil polynomials of K3 surfaces over a fixed finite field $\mathbb{F}_q$, though it is still unknown whether all such polynomials do arise from K3 surfaces defined over $\mathbb{F}_q$. On the other hand, Taelman and Ito prove that under mild hypotheses, every Weil polynomial on such a list is realized by a K3 surface defined over an extension of $\mathbb{F}_q$.

From the Weil conjectures and properties of crystalline cohomology, the Weil polynomials $L_S(T)$ must satisfy the following arithmetic constraints, see [33, 34, 52].

Theorem 3.2. *Let S be a K3 surface over $\mathbb{F}_q$. Then the Weil polynomial $L_S(T) \in \mathbb{Q}[T]$ is a degree 22 polynomial, all of whose roots have complex absolute value 1, and satisfies:*

(1) *Projectivity. $L_S(T)$ has a factor of $1-T$.*
(2) *Weil conjectures. $L_S(T) \in \mathbb{Z}_\ell[T]$ for all ℓ prime to q.*
(3) *Crystalline. Factor $L_S(T) = L_{S,alg}(T)L_{S,trc}(T)$ where $L_{S,alg}(T)$ is the maximal factor all of whose roots are roots of unity. Then either S is supersingular, in which case $L_S(T) = L_{S,alg}(T)$, or $L_{S,trc}(T)$ satisfies the following two properties:*
    (a) *Newton above Hodge. The Newton polygon of $L_{S,trc}(T) \in \mathbb{Q}_p(T)$ lies above the Hodge polygon of the crystalline transcendental lattice of S.*
    (b) *Transcedental. $L_{S,trc}(T) = Q^e$ for some $e > 0$ and $Q \in \mathbb{Q}[T]$ irreducible, where $Q = Q_{<0}Q_{\geq 0}$ in $\mathbb{Q}_p[T]$ where $Q_{<0}$ is irreducible and consists of all factors of Q of the form $(1-\gamma T)$ with $v_p(\gamma) < 0$.*

    We note that when $q = p$, the conditions (a) *and* (b) *are together equivalent to $pL_S(T) \in \mathbb{Z}_p[T]$.*
(4) *Nonnegative point counts. $L_S(T)$ is consistent with $|S(\mathbb{F}_{q^n})| \geq 0$ for all $n \geq 1$.*
(5) *Field extension growth. $L_S(T)$ is consistent with $|S(\mathbb{F}_{q^m})| \geq |S(\mathbb{F}_{q^n})|$ for all $m, n \geq 1$ with $n|m$.*
(6) *Artin–Tate. Writing $L_S(T) = (1-T)^r L_1(T)$ with $L_1(1) \neq 0$, we have $qL_1(-1)$ is a square (possibly 0).*

We say that a degree 22 Weil polynomial satisfying these conditions is of K3 type over $\mathbb{F}_q$. One reasonable question is whether these conditions provide a complete solution to Honda for K3s. To test this, Kedlaya and Sutherland in [33, Computation 3] enumerated the entire list of Weil polynomials of K3 type[1] over $\mathbb{F}_2$, finding 1,672,565 such polynomials. They compare this list to the Weil polynomials of quartic K3 surfaces over $\mathbb{F}_2$, which account for only about 3% of all Weil polynomials of K3 type for $q = 2$. It would be a natural next step to create a census K3 surfaces of other low degree over $\mathbb{F}_2$.

One could also consider a "Honda–Tate for noncommutative K3s" for the larger class of noncommutative K3 surfaces C over a finite field $\mathbb{F}_q$ together with their Weil polynomials $L_{\mathsf{C}}(T)$ introduced in Section 1.1. In this context, formulating "Tate for noncommutative K3s" would require a notion of isogeny between noncommutative K3 surfaces, whose details are not entirely clear, but should at least imply that isogenous noncommutative K3 surfaces have the same Weil polynomials. In order to resolve "Honda for noncommutative K3s" one must formulate a reasonable list of necessary properties satisfied by Weil polynomials of noncommutative K3s, and then ask whether these properties are sufficient.

As a concrete example, we now list some known conditions on the Weil polynomials of K3 categories of cubic fourfolds, in the case $q = p$ for simplicity. To that end, we introduce some formal point counts associated to a noncommutative K3 surface.

[1]The degree 21 polynomial $L(T) \in \mathbb{Z}[T]$ appearing in [33] is expressed in terms of our degree 22 Weil polynomial as $L(T) = qL_S(T/q)/(1-T)$.

In analogy with (1), we define the cubic fourfold point count of C as

$$|X_{\mathsf{C}}(\mathbb{F}_{q^n})| = 1 + q^n|\mathsf{A}_X(\mathbb{F}_{q^n})| + q^{2n} + q^{4n}$$

and in analogy with (3) we define the Hilbert square point count of C as

$$|\mathsf{C}^{[2]}(\mathbb{F}_{q^n})| = \binom{|\mathsf{C}(\mathbb{F}_{q^n})|}{2} + (q^n+1)|\mathsf{C}(\mathbb{F}_{q^n})| + \frac{|\mathsf{C}(\mathbb{F}_{q^{2n}})| - |\mathsf{C}(\mathbb{F}_{q^n})|}{2}$$

Proposition 3.3. *Let C be the K3 category of a cubic fourfold over $\mathbb{F}_p$. Then the Weil polynomial $L_{\mathsf{C}}(T) \in \mathbb{Q}[T]$ is a degree 22 polynomial, all of whose roots have complex absolute value 1, and satisfies:*

(1) *Weil Conjectures and Crystalline: $pL_{\mathsf{C}}(T) \in \mathbb{Z}[T]$.*
(2) *Nonnegative cubic fourfold point counts. $L_{\mathsf{C}}(T)$ is consistent with $|X_{\mathsf{C}}(\mathbb{F}_{p^n})| \geq 0$ for all $n \geq 1$.*
(3) *Cubic fourfold field extension growth. $L_{\mathsf{C}}(T)$ is consistent with $|X_{\mathsf{C}}(\mathbb{F}_{p^m})| \geq |X_{\mathsf{C}}(\mathbb{F}_{q^n})|$ for all $m, n \geq 1$ with $n|m$.*
(4) *Nonnegative Hilbert square point counts. $L_{\mathsf{C}}(T)$ is consistent with $|\mathsf{C}^{[2]}(\mathbb{F}_{p^n})| \geq 0$ for all $n \geq 1$.*
(5) *Hilbert square field extension growth. $L_{\mathsf{C}}(T)$ is consistent with $|\mathsf{C}^{[2]}(\mathbb{F}_{p^m})| \geq |\mathsf{C}^{[2]}(\mathbb{F}_{p^n})|$ for all $m, n \geq 1$ with $n|m$.*
(6) *Artin–Tate. Writing $L_{\mathsf{C}}(T) = (1-T)^r L_1(T)$ with $L_1(1) \neq 0$, we have $pL_1(-1)$ is a square (possibly 0).*

Proof. Parts (1), (2), (3) come from being a factor of the cubic fourfold Weil polynomial. Parts (4) and (5) comes from Proposition 2.9. Part (6) comes from the fact that A_X is the left orthogonal to a exceptional collection, so that $L_{\mathsf{A}_X}(T)$ differs from $L_X(T)$ by factors of $(1-T)$, and then applying [19, Theorem 1.9] for $p > 2$ or [7, Theorem 4.6] for $p = 2$. □

As in the case for K3 surfaces, for a noncommutative K3 surface C, we write $L_{\mathsf{C}}(T) = L_{\mathsf{C},\mathrm{alg}}(T)L_{\mathsf{C},\mathrm{trc}}(T)$ where $L_{\mathsf{C},\mathrm{alg}}(T)$ is the maximal factor all of whose roots are roots of unity. For a K3 surface S, the Tate conjecture (which are proved in [48], [16], [17], [45], [37], [46], [35])) implies that the geometric Picard rank of S equals the degree of $L_{S\mathrm{alg}}$ and the arithmetic Picard rank (i.e., Picard rank over $\mathbb{F}_q$) of S equals the multiplicity of $(1-T)$ in $L_{S,\mathrm{alg}}$. One important difference in the noncommutative setting is that $L_{\mathsf{C},\mathrm{alg}}(T)$ may equal 1; we call such noncommutative K3 surfaces "purely transcendental." Hence by analogy, purely transcendental noncommutative K3 surfaces have "geometric Picard rank 0." For $q = 2$, we can collect some statistics about the distribution of the arithmetic Picard rank ρ (i.e., the multiplicity in $L_{\mathsf{C}}(T)$ of the root 1) and geometric Picard rank $\overline{\rho}$ (i.e., the degree of $L_{\mathsf{C},\mathrm{alg}}(T)$) among Weil polynomials of degree 22 over $\mathbb{F}_q$ that satisfy a minimal set of necessary conditions to be potentially realizable by a noncommutative K3 surface.

Computation 3.4. *There are 5,478,058 degree 22 Weil polynomials satisfying the "Weil conjecture and crystalline" condition from Proposition 3.3 over $\mathbb{F}_2$. Of these, we record the distribution of (geometric) Picard ranks:*

$\overline{\rho}$	0	2	4	6	8	10
#	74846	242700	441072	697944	762944	936736

$\overline{\rho}$	12	14	16	18	22	22
#	775320	651600	442308	270180	122128	60280

ρ	0	1	2	3	4	5
#	2506876	956904	956904	349118	349118	121936
ρ	6	7	8	9	10	11
#	121936	40194	40194	12574	12574	3612
ρ	12	13	14	15	16	17
#	3612	966	966	230	230	48
ρ	18	19	20	21	22	
#	48	8	8	1	1	

We also remark that of the 74,846 purely transcendental Weil polynomials above, only 4,294 come from K3 categories of a cubic fourfolds over $\mathbb{F}_2$.

Remark 3.5. Since roots of $L_{\mathsf{C},\mathrm{trc}}(T)$ must come in complex conjugate pairs, the possible values of $\overline{\rho}$ are even and that for every odd number $n = 1, \ldots, 21$, the number of Weil polynomials with $\rho = n$ and $\rho = n+1$ is the same. This is the same reason why the Tate conjecture implies the well-known fact that the geometric Picard rank of a K3 surface over a finite field must be even.

4. The zeta function of a nonadmissible cubic fourfold

The goal of this section is give a particular example of a Noether–Lefschetz special cubic fourfold X over $\mathbb{Q}$ with nongeometric K3 category. The example suggests that the zeta function alone cannot be used to detect nongeometric K3 categories.

Theorem 4.1. *There exists a cubic fourfold $X/\mathbb{Z}$ such that*

(1) *$X_{\mathbb{Q}}$ has no associated K3 and no associated twisted K3 surface over $\mathbb{C}$ (i.e., $X_{\mathbb{Q}}$ is not admissible nor twisted-admissible over $\mathbb{C}$);*
(2) *X has good reduction at $p = 2$ and the specialization $\mathrm{CH}^2(X_{\overline{\mathbb{Q}}}) \to \mathrm{CH}^2(X_{\overline{\mathbb{F}}_2})$ is an isomorphism; and*
(3) *the point counts of $\mathsf{A}_{X_{\mathbb{F}_2}}$ satisfy the conditions of Theorem 3.2.*

In light of the description of the point counts of $\mathsf{A}_{X_{\mathbb{F}_2}}$ above, it is still plausible that $Z_{\mathsf{A}_{X_{\mathbb{F}_2}}}(T)$ is the zeta function of an actual K3 surface over $\mathbb{F}_2$. Indeed, the main result of Taelman [52] and the computational evidence in [33] suggest that if A_X has point counts resembling a K3 surface then $Z_{\mathsf{A}_X}(T)$ should be equal to the zeta function of an actual K3 surface over k. Therefore, the results of this section can be taken as evidence that point-counting on a K3 category cannot alone distinguish geometric K3 categories from nongeometric ones.

4.1. A special cubic fourfold over $\mathbb{Q}$ with no (twisted) admissible markings.

Let d be a positive integer congruent to 0 or 2 modulo 6. We say that d is an *admissible discriminant* if d satisfies the condition

$$4 \nmid d, 9 \nmid d, \text{ and } p \nmid d \text{ for any odd prime } p \equiv 2 \mod 3 \quad (\star\star).$$

We say that d is a *twisted admissible discriminant* if there is some integer k such that

$$d = k^2 d_0 \text{ for some admissible discriminant } d_0 \quad (\star\star').$$

For a cubic fourfold X over the complex numbers, Huybrechts [31] has shown, following closely-related work of Addington and Thomas [4], that $\mathrm{CH}^2(X)$ contains a primitive rank 2 sublattice K of *(twisted) admissible discriminant* d with

$c_1(\mathcal{O}_X(1))^2 \in K$ if and only if $\mathsf{A}_X \cong \mathsf{D}^{\mathrm{b}}(S,\alpha)$ for some twisted K3 surface (S,α). We in turn say that the primitive sublattice K is a *(twisted) admissible* sublattice.

Proposition 4.2. *If X is a cubic fourfold $X/\mathbb{C}$ with $\operatorname{rk}\mathrm{CH}^2(X) = 3$ and X contains a Veronese surface V and a cubic scroll T such that $T.V = 2$, then A_X is nongeometric; that is, A_X is not equivalent to the derived category of a (twisted) K3 surface.*

Proof. Let $h = c_1(\mathcal{O}_X(1))$. By the hypotheses, the cubic fourfold X contains a cubic scroll, so X has a rank 2 marking of discriminant 12, and a discriminant 20 marking as well because it contains a Veronese [26]. Since $T.V = 2$, the lattice $\langle h^2, T, V\rangle \subset \mathrm{CH}^2(X)$ has Gram matrix

$$\begin{pmatrix} 3 & 4 & 3 \\ 4 & 12 & 2 \\ 3 & 2 & 7 \end{pmatrix}$$

We note that in [54, §8], Yang and Yu give a classification of rank 3 positive definite lattices M that can contain (twisted) admissible primitive sublattices. As a consequence of their classification, M has *no admissible* sublattices. We wish to show the stronger claim that M has no *twisted admissible* primitive sublattices. Indeed, the discriminant of any rank 2 primitive sublattice containing h^2 of the rank 3 lattice above is generically of the form $d = 12y(y-z) + 20z^2$ for integers y and z and so cannot be admissible (since $4 \mid d$), and further more cannot be twisted admissible: if d were twisted admissible, then $d/4 = 3y(y-z) + 5z^2$ would need to be of the form $s^2 d_0$ for some admissible d_0, so in particular we would need $d/4 \equiv 0$ or 2 modulo 6, but $3y(y-z)+5z^2$ never represents these congruence classes. □

Remark 4.3. In Proposition 4.2, we are using a very general cubic fourfold contained in the intersection $\mathcal{C}_{12} \cap \mathcal{C}_{20}$ of Hassett divisors in the moduli space of cubic fourfolds. In fact, it appears that $\mathcal{C}_{12} \cap \mathcal{C}_{20}$ has six irreducible components, determined by the possible intersection numbers $T.V = -1, \ldots, 4$. We utilize the component where $T.V = 2$, and recently, the geometry of the component with $T.V = 4$ has been investigated in [51].

Proof of Theorem 4.1. Let $X/\mathbb{Z}$ be the cubic fourfold with equation

$$\begin{aligned}
&-27195x_1^3+99309x_1^2x_2+52143x_1^2x_3-19299x_1^2x_4+17717x_1^2x_5-166089x_1^2x_6+280203x_1x_2^2+42138x_1x_2x_3-24486x_1x_2x_4+335080x_1x_2x_5 \\
&+36287x_1x_2x_6-52038x_1x_3^2+42628x_1x_3x_4-91243x_1x_3x_5+76026x_1x_3x_6+74191x_1x_4^2+105644x_1x_4x_5-206488x_1x_4x_6 \\
&-21765x_1x_5^2-396946x_1x_5x_6-145953x_1x_6^2+153699x_2^3-17064x_2^2x_3-12246x_2^2x_4+317363x_2^2x_5+202376x_2^2x_6-45743x_2x_3^2+76777x_2x_3x_4 \\
&\quad -160450x_2x_3x_5-622x_2x_3x_6+102045x_2x_4^2+105638x_2x_4x_5-206876x_2x_4x_6+104046x_2x_5^2-97682x_2x_5x_6 \\
&+48677x_2x_6^2+27090x_3^3+54944x_3^2x_4-93628x_3^2x_5-11594x_3^2x_6+27854x_3x_4^2-13462x_3x_4x_5+97190x_3x_4x_6-149681x_3x_5^2-126562x_3x_5x_6 \\
&+44296x_3x_6^2+74191x_4^2x_5+102045x_4^2x_6+97089x_4x_5^2-76364x_4x_5x_6-194630x_4x_6^2-52559x_5^3-376318x_5^2x_6-296287x_5x_6^2x_1 = 0.
\end{aligned}$$

We wish to show that X has the claimed properties (1) and (2). We first show that $X_{\mathbb{Q}}$ contains surfaces T and V as in the statement of Proposition 4.2. The cubic X contains the cubic scroll T given by the simultaneous vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} x_5 & x_2 + x_3 + x_6 \\ x_5 + x_6 & x_2 + x_6 \\ x_3 + x_4 + x_6 & x_5 \end{pmatrix},$$

inside of the hyperplane $x_1 + x_2 + x_5 + x_6 = 0$. One also computes that X contains the Veronese surface V given by the vanishing of the minors of the matrix

$$\begin{pmatrix} x_3 + x_5 & x_1 + x_2 + x_4 & x_3 + x_4 \\ x_1 + x_2 + x_4 & x_1 + x_4 + x_6 & x_3 + x_6 \\ x_3 + x_4 & x_3 + x_6 & x_1 + x_2 + x_5 + x_6 \end{pmatrix}$$

The Magma code [14] in the arXiv distribution of this article verifies these claims and the claim that $T.V = 2$. We thus know that $\operatorname{rk}\mathrm{CH}^2(X_{\mathbb{Q}}) \geq 3$.

We next verify using Magma that the reduction $X_{\mathbb{F}_2}$ is smooth. Now, for any $\ell \neq 2$, we can compute via the point counting algorithm of [7, Section 4.2] that the primitive Weil-polynomial $f(t) = \det(F^* - t\mathrm{Id}|H^4_{\text{ét,pr}}(X_{\overline{\mathbb{F}}_2}, \mathbb{Q}_\ell(2)))$ of $X_{\mathbb{F}_2}$, is given by

$$\begin{aligned} f(t) =& t^{22} - t^{21} + t^{20} - \frac{3}{2}t^{19} + t^{18} - \frac{3}{2}t^{17} + \frac{3}{2}t^{16} - t^{15} + 2t^{14} - 2t^{13} + \frac{3}{2}t^{12} \\ & - 2t^{11} + \frac{3}{2}t^{10} - 2t^9 + 2t^8 - t^7 + \frac{3}{2}t^6 - \frac{3}{2}t^5 + t^4 - \frac{3}{2}t^3 + t^2 - t + 1. \end{aligned}$$

This Weil polynomial factors as $f(t) = (t-1)^2 g(t)$ for an irreducible, noncyclotomic polynomial of degree 20 over $\mathbb{Q}$. It follows from the Tate conjecture for cubic fourfolds over $\mathbb{F}_2$ (see [7, Section 4.6]) that $\operatorname{rk}\mathrm{CH}^2(X_{\overline{\mathbb{F}}_2}) = 3$. By the specialization theorem for Chow groups [22], cf. [1, §2], one has

$$3 = \operatorname{rk}\mathrm{CH}^2(X_{\overline{\mathbb{F}}_2}) \geq \operatorname{rk}\mathrm{CH}^2(X_{\overline{\mathbb{Q}}}).$$

Rigidity for Chow groups, together with the fact that they are torsion-free, shows that extension of scalars yields an isometry $\mathrm{CH}^2(X_{\overline{\mathbb{Q}}}) \cong \mathrm{CH}^2(X_{\mathbb{C}})$. This implies that the latter group has rank 3, and that by Proposition 4.2, we have that $X_{\mathbb{Q}}$ has no twisted associated K3 surfaces (over $\mathbb{C}$).

It remains to check the that point counts of the K3 category of $X_{\mathbb{F}_2}$ are positive and exhibit expected growth. By the formula (4), it is enough to confirm the point count growth properties over $\mathbb{F}_{2^k}$, $k = 1, \ldots, 4$ (cf. [33]), but we present a bit more:

k	1	2	3	4	5	6	7	8	9	10	11
$\lvert\mathsf{A}_{X_{\mathbb{F}_2}}(\mathbb{F}_{2^k})\rvert$	7	13	85	273	1137	4081	16289	64001	264001	1052673	4196353

Hence the point counts of $\mathsf{A}_{X_{\mathbb{F}_2}}$ satisfy (3). $\square$

Remark 4.4. To find the example we provided above, we used our complete tabulation of zeta functions of cubic fourfolds over $\mathbb{F}_2$ from [7] to find a cubic with the desired algebraic and geometric rank of 3 and with discriminant 68 according to the (conjectural) Artin-Tate formula for fourfolds [42]. We then verified that this cubic X' over $\mathbb{F}_2$ contains a configuration of a Veronese and a cubic scroll intersecting in two points. Finally, we found a smooth lift $X/\mathbb{Q}$ containing the a Veronese and a scroll in the right configuration by first lifting the Veronese and scroll from $\mathbb{F}_2$ first, and then searching for lifts of X' containing these two surfaces.

Remark 4.5.

(1) Since one expects something like Proposition 4.2 to hold in positive characteristic, it is conceivable that the cubic fourfolds in Theorem 4.1 have nongeometric K3 category over $\overline{\mathbb{F}}_2$.
(2) However, the Weil polynomial of $\mathsf{A}_{X_{\mathbb{F}_2}}$ are of K3 type, and so appear on Kedlaya and Sutherland's list of potential Weil polynomials of K3 surfaces over $\mathbb{F}_2$ [33, Computation 3(c)]. This suggests that nongeometric K3 categories

may have point counts identical to those of K3 surfaces, or that additional restrictions on the Weil polynomials of K3 surfaces are needed for a complete Honda–Tate theory for K3 surfaces.

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