FAILURE OF THE LOCAL-GLOBAL PRINCIPLE FOR ISOTROPY OF QUADRATIC FORMS OVER RATIONAL FUNCTION FIELDS

ASHER AUEL AND V. SURESH

Abstract. We prove the failure of the local-global principle, with respect to all discrete valuations, for isotropy of quadratic forms of dimension 2^n over a rational function field of transcendence degree n over $\mathbb{C}$. Our construction involves the generalized Kummer varieties considered by Borcea [6] and Cynk and Hulek [11].

Introduction

The Hasse–Minkowski theorem states that if a quadratic form q over a number field is isotropic over every completion, then q is isotropic. This is the first, and most famous, instance of the *local-global principle* for isotropy of quadratic forms. Already for a rational function field in one variable over a number field, Witt [20] found examples of the failure of the local-global principle for isotropy of quadratic forms in dimension 3 (and also 4). Lind [17] and Reichardt [18], and later Cassels [7], found examples of failure of the local-global principle for isotropy of pairs of quadratic forms of dimension 4 over $\mathbb{Q}$ (see [1] for a detailed account), giving examples of quadratic forms over $\mathbb{Q}(t)$ by an application of the Amer–Brumer theorem [2], [13, Thm. 17.14]. Cassels, Ellison, and Pfister [8] found examples of dimension 4 over a rational function field in two variables over the real numbers.

Here, we are interested in the failure of the local-global principle for isotropy of quadratic forms over function fields of higher transcendence degree over algebraically closed fields. All our fields will be assumed to be of characteristic $\neq 2$ and all our quadratic forms nondegenerate. Recall that a quadratic form is isotropic if it admits a nontrivial zero. If K is a field and v is a discrete valuation on K, we denote by K_v the fraction field of the completion (with respect to the v-adic topology) of the valuation ring of v. When we speak of the local-global principle for isotropy of quadratic forms, sometimes referred to as the Strong Hasse Principle, in a given dimension d over a given field K, we mean the following statement:

> If q is a quadratic form in d variables over K and q is isotropic over K_v for every discrete valuation v on K, then q is isotropic over K.

Our main result is the following.

Theorem 1. *Fix any $n \geq 2$. The local-global principle for isotropy of quadratic forms fails to hold in dimension 2^n over the rational function field $\mathbb{C}(x_1, \ldots, x_n)$.*

Previously, only the case of $n = 2$ was known, with the first known explicit examples appearing in [15], and later in [5] and [14]. For a construction, using algebraic geometry, over any transcendence degree 2 function field over an algebraically closed field of characteristic 0, see [3], [4, §6]. This later result motivates the following.

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Conjecture 2. *Let K be a finitely generated field of transcendence degree $n \geq 2$ over an algebraically closed field k of characteristic $\neq 2$. Then the local-global principle for isotropy of quadratic forms fails to hold in dimension 2^n over K.*

We recall that by Tsen–Lang theory [16, Theorem 6], such a function field is a C_n-field, hence has u-invariant 2^n, and thus all quadratic forms of dimension $> 2^n$ are already isotropic. An approach to Conjecture 2, along the lines of the proof in the $n = 2$ case given in [4, Cor. 6.5], is outlined in Section 4.

Finally, we point out that in the $n = 1$ case, with $K = k(X)$ for a smooth projective curve X over an algebraically closed field k, the local-global principle for isotropy of binary quadratic forms (equivalent to the "global square theorem") holds when X has genus 0 and fails for X of positive genus.

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1. Hyperbolicity over a quadratic extension

Let K be a field of characteristic $\neq 2$. We will need the following result about isotropy of quadratic forms, generalizing a well-known result in the dimension 4 case, see [19, Ch. 2, Lemma 14.2].

Proposition 1.1. *Let $n > 0$ be divisible by four, q a quadratic form of dimension n and discriminant d defined over K, and $L = K(\sqrt{d})$. If q is hyperbolic over L then q is isotropic over K.*

Proof. If $d \in K^{\times 2}$, then $K = K(\sqrt{d})$ and hence q is hyperbolic over K. Suppose $d \notin K^{\times 2}$ and q is anisotropic. Since q_L is hyperbolic, $q \simeq \langle 1, -d \rangle \otimes q_1$ for some quadratic form q_1 over K, see [19, Ch. 2, Theorem 5.3]. Since the dimension of q is divisible by four, the dimension of q_1 is divisible by two, and a computation of the discriminant shows that $d \in K^{\times 2}$, which is a contradiction. □

For $n \geq 1$ and $a_1, \dots, a_n \in K^\times$, recall the n-fold Pfister form

$$\langle\langle a_1, \dots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$$

and the associated symbol $(a_1) \cdots (a_n)$ in the Galois cohomology group $H^n(K, \mu_2^{\otimes n})$. Then $\langle\langle a_1, \dots, a_n \rangle\rangle$ is hyperbolic if and only if $\langle\langle a_1, \dots, a_n \rangle\rangle$ is isotropic if and only if $(a_1) \cdots (a_n)$ is trivial.

For $d \in K^\times$ and $n \geq 2$, we will consider quadratic forms of discriminant d related to n-fold Pfister forms, as follows. Write $\langle\langle a_1, \dots, a_n \rangle\rangle$ as $q_0 \perp \langle (-1)^n a_1 \dots a_n \rangle$, then define $\langle\langle a_1, \dots, a_n; d \rangle\rangle = q_0 \perp \langle (-1)^n a_1 \dots a_n d \rangle$. For example:

$$\langle\langle a, b; d \rangle\rangle = \langle 1, -a, -b, abd \rangle$$
$$\langle\langle a, b, c; d \rangle\rangle = \langle 1, -a, -b, -c, ab, ac, bc, -abcd \rangle$$

for $n = 2$ and $n = 3$, respectively. We remark that every quadratic form of dimension 4 is similar to one of this type. If $q = \langle\langle a_1, \dots, a_n; d \rangle\rangle$, we note that, in view of Proposition 1.1 and the fact that q_L is a Pfister form over $L = K(\sqrt{d})$, we have that q is isotropic if and only if q_L is isotropic, generalizing a well-known result about quadratic forms of dimension 4, see [19, Ch. 2, Lemma 14.2].

2. Generalized Kummer varieties

In this section, we review a construction, considered in the context of modular Calabi–Yau varieties [11, §2] and [12], of a generalized Kummer variety attached to a product of elliptic curves. This recovers, in dimension 2, the Kummer K3 surface associated to a decomposable abelian surface, and in dimension 3, a class of Calabi–Yau threefolds of CM type considered by Borcea [6, §3].

Let $E_1, \dots, E_n$ be elliptic curves over an algebraically closed field k of characteristic $\neq 2$ and let $Y = E_1 \times \cdots \times E_n$. Let σ_i denote the negation automorphism on E_i and $E_i \to \mathbb{P}^1$ the associated quotient branched double cover. We lift each σ_i to an automorphism of Y; the subgroup $G \subset \mathrm{Aut}(Y)$ they generate is an elementary abelian 2-group. Consider the exact sequence of abelian groups

$$1 \to H \to G \xrightarrow{\Pi} \mathbb{Z}/2 \to 0,$$

where Π is defined by sending each σ_i to 1. Then the product of the double covers $Y \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is the quotient by G and we denote by $Y \to X$ the quotient by the subgroup H. Then the intermediate quotient $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is a double cover, branched over a reducible divisor of type $(4, \dots, 4)$.

We point out that X is a singular degeneration of smooth Calabi–Yau varieties that also admits a smooth Calabi–Yau model, see [11, Cor. 2.3]. For $n = 2$, the minimal resolution of X is isomorphic to the Kummer K3 surface $\mathrm{Kum}(E_1 \times E_2)$.

Given nontrivial classes $\gamma_i \in H^1_{\text{ét}}(E_i, \mu_2)$, we consider the cup product

$$\gamma = \gamma_1 \cdots \gamma_n \in H^n_{\text{ét}}(Y, \mu_2^{\otimes n})$$

and its restriction to the generic point of Y, which is a class in the unramified part $H^n_{\mathrm{nr}}(k(Y)/k, \mu_2^{\otimes n})$ of the Galois cohomology group $H^n(k(Y), \mu_2^{\otimes n})$ of the function field $k(Y)$ of Y (see [9] for background on the unramified cohomology groups). These classes have been studied in [10]. We remark that γ is in the image of the restriction map $H^n(k(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1), \mu_2^{\otimes n}) \to H^n(k(Y), \mu_2^{\otimes n})$ in Galois cohomology since each γ_i is in the image of the restriction map $H^1(k(\mathbb{P}^1), \mu_2) \to H^1(k(E_i), \mu_2)$.

We make this more explicit as follows. For each double cover $E_i \to \mathbb{P}^1$, we choose a Weierstrass equation in Legendre form

$$y_i^2 = x_i(x_i - 1)(x_i - \lambda_i) \tag{1}$$

where x_i is a coordinate on $\mathbb{P}^1$ and $\lambda_i \in k \smallsetminus \{0, 1\}$. Then the branched double cover $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is birationally defined by the equation

$$y^2 = \prod_{i=1}^{n} x_i(x_i - 1)(x_i - \lambda_i) = f(x_1, \dots, x_n) \tag{2}$$

where $y = y_1 \cdots y_n$ in $\mathbb{C}(Y)$. Up to an automorphism, we can even choose the Legendre forms so that the image of γ_i under $H^1_{\text{ét}}(E, \mu_2) \to H^1(k(E), \mu_2)$ coincides with the square class (x_i) of the function x_i, which clearly comes from $k(\mathbb{P}^1)$.

The main result of this section is that the class γ considered above is already unramified over $k(X)$. We prove a more general result.

Proposition 2.1. *Let k be an algebraically closed field of characteristic $\neq 2$ and $K = k(x_1, \dots, x_n)$ a rational function field over k. For $1 \leq i \leq n$, let $f_i(x_i) \in k[x_i]$ be polynomials of even degree satisfying $f_i(0) \neq 0$, and let $f = \prod_{i=1}^n x_i f(x_i)$. Then the restriction of the class $\xi = (x_1) \cdots (x_n) \in H^n(K, \mu_2^{\otimes n})$ to $H^n(K(\sqrt{f}), \mu_2^{\otimes n})$ is unramified with respect to all discrete valuations.*

Proof. Let $L = K(\sqrt{f})$ and v a discrete valuation on L with valuation ring $\mathcal{O}_v$, maximal ideal $\mathfrak{m}_v$, and residue field $k(v)$.

Suppose $v(x_i) < 0$ for some i. Let d_i be the degree of f_i and consider the reciprocal polynomial $f_i^*(x_i) = x_i^{d_i} f_i(\frac{1}{x_i})$, so that $x_i f_i(x_i) = x^{d_i+2} \cdot \frac{1}{x_i} f_i^*(\frac{1}{x_i})$. Since d_i is even, we have that the polynomials $x_i f_i(x_i)$ and $\frac{1}{x_i} f_i^*(\frac{1}{x_i})$ have the same class in $K^\times/K^{\times 2}$.

Thus, up to replacing, for all i with $v(x_i) < 0$, the polynomial f_i by f_i^* in the definition of f and replacing x_i by $\frac{1}{x_i}$, we can assume that $v(x_i) \geq 0$ for all i without changing the extension L/K. Hence $k[x_1, \ldots, x_n] \subset \mathcal{O}_v$.

Let $\mathfrak{p} = k[x_1, \ldots, x_n] \cap \mathfrak{m}_v$. Then $\mathfrak{p}$ is a prime ideal of $k[x_1, \ldots, x_n]$ whose residue field is a subfield of $k(v)$. Let $K_\mathfrak{p}$ be the completion of K at $\mathfrak{p}$ and L_v the completion of L at v. Then $K_\mathfrak{p}$ is a subfield of L_v.

If $v(x_i) = 0$ for all i, then ξ is unramified at v. Suppose that $v(x_i) \neq 0$ for some i. By reindexing $x_1, \ldots, x_n$, we assume that there exists $m \geq 1$ such that $v(x_i) > 0$ for $1 \leq i \leq m$ and $v(x_i) = 0$ for $m+1 \leq i \leq n$, i.e., $x_1, \ldots, x_m \in \mathfrak{p}$ and $x_{m+1}, \ldots, x_n \notin \mathfrak{p}$. In particular, the transcendence degree of $k(\mathfrak{p})$ over k is at most $n - m$.

First, suppose $f_i(x_i) \in \mathfrak{p}$ for some $m + 1 \leq i \leq n$. Since $f_i(x_i)$ is a product of linear factors in $k[x_i]$, we have that $x_i - a_i \in \mathfrak{p}$ for some $a_i \in k$, with $a_i \neq 0$ since $f_i(0) \neq 0$. Thus the image of x_i in $k(\mathfrak{p})$ is equal to a_i and hence a square in $K_\mathfrak{p}$. In particular, x_i is a square in L_v, thus ξ is trivial (hence unramified) at v.

Now, suppose that $f_i(x_i) \notin \mathfrak{p}$ for all $m + 1 \leq i \leq n$. Then for each $1 \leq i \leq m$, we see that since $x_i \in \mathfrak{p}$ and $f_i(0) \neq 0$, we have $f_i(x_i) \notin \mathfrak{p}$. Consequently, we can assume that $f = x_1 \cdots x_m u$ for some $u \in k[x_1, \ldots, x_n] \smallsetminus \mathfrak{p}$.

Computing with symbols, we have

$$(x_1) \cdots (x_m) = (x_1) \cdots (x_{m-1}) \cdot (x_1 \cdots x_m) \in H^m(K, \mu_2^{\otimes m}).$$

By definition, $f = x_1 \cdots x_m u$ is a square in L, and thus we have that

$$(x_1) \cdots (x_m) = (x_1) \cdots (x_{m-1}) \cdot (u) \in H^m(L, \mu_2).$$

Thus it is enough to show that $(x_1) \cdots (x_{m-1}) \cdot (u) \cdot (x_{m+1}) \cdots (x_n)$ is unramified at v.

Let $\partial_v : H^n(L, \mu_2^{\otimes n}) \to H^{n-1}(k(v), \mu_2^{\otimes n-1})$ be the residue homomorphism at v. Since x_i, for all $m + 1 \leq i \leq n$, and u are units at v, we have

$$\partial_v((x_1) \cdots (x_{m-1}) \cdot (u) \cdot (x_{m+1}) \cdots (x_n)) = \alpha \cdot (\overline{u}) \cdot (\overline{x}_{m+1}) \cdots (\overline{x}_n)$$

for some $\alpha \in H^{m-2}(k(v), \mu_2^{m-2})$, where for any $h \in k[x_1, \ldots, x_n]$, we write $\overline{h}$ for the image of h in $k(\mathfrak{p})$. Since the transcendence degree of $\kappa(\mathfrak{p})$ over k is at most $n - m$ and k is algebraically closed, we have that $H^{n-m+1}(\kappa(\mathfrak{p}), \mu_2^{\otimes n-m+1}) = 0$. Since $\overline{u}, \overline{x}_i \in \kappa(\mathfrak{p})$, we have $(\overline{u}) \cdot (\overline{x}_{m+1}) \cdots (\overline{x}_n) = 0$. In particular $\partial_v(\xi) = 0$. Finally, the class ξ is unramified at all discrete valuations on L. □

As an immediate consequence, we have the following.

Proposition 2.2. *Let $E_1, \ldots, E_n$ be elliptic curves over an algebraically closed field k of characteristic $\neq 2$, given in the Legendre form (1), with $K = k(x_1, \ldots, x_n)$. Then the restriction of the class $\gamma = (x_1) \cdots (x_n)$ in $H^n(K, \mu_2^{\otimes n})$ to $H^n(k(X), \mu_2^{\otimes n})$ is unramified at all discrete valuations.*

Finally, we will need the fact, proved in the appendix by Gabber to the article [10], that if $k = \mathbb{C}$ and the j-invariants $j(E_1), \ldots, j(E_n)$ are algebraically independent, then any cup product class $\gamma = \gamma_1 \cdots \gamma_n \in H^n(\mathbb{C}(Y), \mu_2^{\otimes n})$, with $\gamma_i \in H^1(\mathbb{C}(E_i), \mu_2)$ nontrivial as considered above, is itself nontrivial.

3. Failure of the local global principle

Given elliptic curves $E_1, \dots, E_n$ defined over $\mathbb{C}$ with algebraically independent j-invariants, presented in Legendre form (1), and $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ the double cover defined by $y^2 = f(x_1, \dots, x_n)$ in (2), we consider the quadratic form

$$q = \ll x_1, \dots, x_n; f \gg$$

over $\mathbb{C}(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1) = \mathbb{C}(x_1, \dots, x_n)$, as in Section 1.

Our main result is that q shows the failure of the local-global principle for isotropy, with respect to all discrete valuations, for quadratic forms of dimension 2^n over $\mathbb{C}(x_1, \dots, x_n)$, thereby proving Theorem 1.

Theorem 3.1. *The quadratic form $q = \ll x_1, \dots, x_n; f \gg$ is anisotropic over $\mathbb{C}(x_1, \dots, x_n)$ yet is isotropic over the completion at every discrete valuation.*

Proof. Let $K = \mathbb{C}(x_1, \dots, x_n)$ and $L = K(\sqrt{f}) = \mathbb{C}(X)$. Let v be a discrete valuation of K and w an extension to L, with completions K_v and L_w and residue fields $\kappa(v)$ and $\kappa(w)$, respectively. We note that $\kappa(v)$ and $\kappa(w)$ have transcendence degree $0 \le i \le n-1$ over $\mathbb{C}$. By Proposition 1.1, we have that $q \otimes_K K_v$ is isotropic if and only if $q \otimes_K L_w$ is isotropic.

By Proposition 2.2, the restriction $(x_1) \cdots (x_n) \in H^n(L, \mu_2^{\otimes n})$ is unramified at w, hence $q \otimes_K L = \ll x_1, \dots, x_n \gg$ is an n-fold Pfister form over L unramified at w. Thus the first residue form for $q \otimes_K L$, with respect to the valuation w, is isotropic since the residue field $\kappa(w)$ is a C_i-field and q has dimension $2^n > 2^i$. Consequently, by a theorem of Springer [19, Ch. 6, Cor. 2.6], $q \otimes_K L$, and thus q, is isotropic.

Finally, q is anisotropic since the symbol $(x_1) \cdots (x_n)$ is nontrivial when restricted to $\mathbb{C}(Y)$ by [10, Appendice], hence is nontrivial when restricted to $\mathbb{C}(X)$. □

To give an explicit example, let $\lambda, \kappa, \nu \in \mathbb{C} \smallsetminus \{0, 1\}$ be algebraically independent complex numbers. Then over the function field $K = \mathbb{C}(x, y, z)$, the quadratic form

$$q = \langle 1, x, y, z, xy, xz, yz, (x-1)(y-1)(z-1)(x-\lambda)(y-\kappa)(z-\nu) \rangle$$

is isotropic over every completion K_v associated to a discrete valuation v of K, and yet q is anisotropic over K.

4. Over general function fields

We have exhibited locally isotropic but globally anisotropic quadratic forms of dimension 2^n over the rational function field $\mathbb{C}(x_1, \dots, x_n)$. In [4, Cor. 6.5], we proved that locally isotropic but anisotropic quadratic forms of dimension 4 exist over any function field of transcendence degree 2 over an algebraically closed field of characteristic zero. Taking these as motivation, we recall Conjecture 2, that over any function field of transcendence degree $n \ge 2$ over an algebraically closed field of characteristic $\ne 2$, there exist locally isotropic but anisotropic quadratic forms of dimension 2^n. In this section, we provide a possible approach to Conjecture 2, motivated by the geometric realization result in [4, Proposition 6.4].

Proposition 4.1. *Let $K = k(X)$ be the function field of a smooth projective variety X of dimension $n \ge 2$ over an algebraically closed field k of characteristic $\ne 2$. If either $H^n_{\mathrm{nr}}(K/k, \mu_2^{\otimes n}) \ne 0$ or $H^n_{\mathrm{nr}}(L/k, \mu_2^{\otimes n}) \ne 0$ for some separable quadratic extension L/K, then there exists an anisotropic quadratic form of dimension 2^n over K that is isotropic over the completion at every discrete valuation.*

Proof. First, by a standard application of the Milnor conjectures, every element in $H^n(K, \mu_2^{\otimes n})$ is a symbol since K is a C_n-field. If $H^n_{\mathrm{nr}}(K/k, \mu_2^{\otimes n}) \neq 0$, then taking a nontrivial element $(a_1)\cdots(a_n)$, the n-fold Pfister form $\ll a_1, \dots, a_n \gg$ is locally isotropic (by the same argument as in the proof of Theorem 3.1) but is anisotropic, giving an example. So we can assume that $H^n_{\mathrm{nr}}(K/k, \mu_2^{\otimes n}) = 0$.

Now assume that $H^n_{\mathrm{nr}}(L/k, \mu_2^{\otimes n}) \neq 0$ for some separable quadratic extension $L = K(\sqrt{d})$ of K. Then taking a nontrivial element $(a_1)\cdots(a_n)$, the corestriction map $H^n_{\mathrm{nr}}(L/k, \mu_2^{\otimes n}) \to H^n_{\mathrm{nr}}(K/k, \mu_2^{\otimes n}) = 0$ is trivial, so by the restriction-corestriction sequence for Galois cohomology, we have that $(a_1)\cdots(a_n)$ is in the image of the restriction map $H^n_{\mathrm{nr}}(K/k, \mu_2^{\otimes n}) \to H^n_{\mathrm{nr}}(L/k, \mu_2^{\otimes n}) = 0$, in which case we can assume that $a_1, \dots, a_n \in K^\times$. Then the quadratic form $\ll a_1, \dots, a_n; d \gg$ is locally isotropic over K (by the same argument as in the proof of Theorem 3.1) but globally anisotropic. □

Hence we are naturally led to the following geometric realization conjecture for unramified cohomology classes.

Conjecture 4.2. *Let K be a finitely generated field of transcendence degree n over an algebraically closed field k of characteristic $\neq 2$. Then either $H^n_{\mathrm{nr}}(K/k, \mu_2^{\otimes n}) \neq 0$ or their exists a quadratic extension L/K such that $H^n_{\mathrm{nr}}(L/k, \mu_2^{\otimes n}) \neq 0$.*

Proposition 4.1 says that the geometric realization Conjecture 4.2 implies Conjecture 2 on the failure of the local-global principle for isotropy of quadratic forms. Proposition 2.2 establishes the conjecture in the case when K is purely transcendental over k; in [4, Proposition 6.4], we established the conjecture in dimension 2 and characteristic 0, specifically, that given any smooth projective surface S over an algebraically closed field of characteristic zero, there exists a double cover $T \to S$ with T smooth and $H^2_{\mathrm{nr}}(k(T)/k, \mu_2^{\otimes 2}) = \mathrm{Br}(T)[2] \neq 0$. In this latter case, Proposition 4.1 gives a different proof, than the one presented in [4, §6], that there exist locally isotropic but anisotropic quadratic forms of dimension 4 over $K = k(S)$.

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Asher Auel, Department of Mathematics, Yale University, New Haven, Connecticut
E-mail address: asher.auel@yale.edu

V. Suresh, Department of Mathematics & Computer Science, Emory University, Atlanta, Georgia
E-mail address: suresh.venapally@emory.edu