A CYCLOTOMIC ARTIN–TATE FORMULA AND SPHERE PACKING FOR K3 SURFACES

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Abstract. We give new necessary conditions on the Weil polynomial of a K3 surface over a finite field using techniques from the theory of sphere packing. By comparing the Artin–Tate formula over a finite extension, we derive a special values formula at arbitrary roots of unity that involves the discriminant of the Frobenius coinvariant Néron–Severi lattice. Known bounds on the minimum discriminants of even rootless lattices then imply nontrivial lower bounds on these special values. Applying these new conditions to Kedlaya and Sutherland's list of candidate Weil polynomials of K3 surfaces over $\mathbb{F}_2$ rules out over 50% of the candidates.

Introduction

Complex abelian varieties and K3 surfaces are controlled by Hodge theory, and they both enjoy well-understood period domains and Torelli theorems. Over a finite field, our analogous understanding of the possible ℓ-adic cohomology groups diverges between the class of abelian varieties and K3 surfaces. Abelian varieties have a well-understood Honda–Tate theory [21, 43], in which isogeny classes are controlled by the characteristic polynomial of Frobenius, and the possible such characteristic polynomials are realized once they satisfy standard necessary conditions imposed by the Weil conjectures and crystalline cohomology. By contrast, an analogous Honda–Tate theory for K3 surfaces over $\mathbb{F}_q$, which attempts to characterize the possible Weil polynomials $P_X(t) = q\det(1 - tF \mid H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)))$ of K3 surfaces X over a finite field, is far from complete. Such a theory was initiated by Elsenhans and Jahnel [15, 16] (who attribute it to Katz) and developed by Kedlaya and Sutherland [25], with further work in different directions by Taelman [41, 42], Ito [23], and the first author with Petok [3].

Besides the necessary conditions imposed by the Weil conjectures and crystalline cohomology, the possible Weil polynomials are constrained by point count nonnegativity and field extension growth [16, 25], as well as by the arithmetic of special values as shown by Elsenhans and Jahnel [16, 17] using the Artin–Tate formula under base change. An investigation of the role of noncommutative K3 surfaces [3] suggests that these necessary conditions are not sufficient.

In this work, we construct a new class of necessary conditions on the Weil polynomial of a K3 surface, derived from techniques in the theory of sphere packing applied to the Frobenius coinvariant sublattice of the geometric Néron–Severi group. Explicitly, this is equivalent to a bound on special values of the Weil polynomial at roots of unity. For simplicity, we present the condition in the case of special values at -1, see Section 2 for the general case.

Theorem 1. *Let X be a K3 surface over $\mathbb{F}_q$ with Weil polynomial $P_X(t)$. Then we have the following quadratic Artin–Tate formula*

$$\lim_{t\to -1} \frac{P_X(t)}{(1+t)^{\rho_-}} = 2^{\rho_2} \frac{|\operatorname{Br}(X_2)|}{|\operatorname{Br}(X)|} \frac{|\operatorname{NS}(X_2)|}{|\operatorname{NS}(X)|}.$$

where $X_2 = X_{\mathbb{F}_{q^2}}$, $\rho = \operatorname{rk}\operatorname{NS}(X)$, and $\rho_2 = \operatorname{rk}\operatorname{NS}(X_2) = \rho + \rho_-$. Explicitly, write

$$P_X(t) = (1-t)^{\rho}(1+t)^{\rho_-}P_2(t),$$

with $P_2(t) \in \mathbb{Z}[t]$ such that $P_2(1)P_2(-1) \neq 0$. If $\rho_- = 0$, then $P_2(-1)$ is a perfect square. If $\rho_- > 0$, then

$$2^{\min(\rho,\rho^-)}\, P_2(-1) \geq \delta_{\rho_-}$$

where δ_k denotes the least determinant of an even positive definite lattice of rank k containing no vector of norm 2.

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We remark that ρ^- is the dimension of the (-1)-eigenspace of Frobenius acting on the ℓ-adic cohomology $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$, and coincides with the Picard rank jump from base changing to $\mathbb{F}_{q^2}$. When the Picard rank does not jump ($\rho_- = 0$), the statement was established by Elsenhans and Jahnel [16]. The new condition is nontrivial precisely in the complementary case, when the Picard rank does jump over $\mathbb{F}_{q^2}$. Bounds on the invariant δ_k, which is important in the theory of sphere packing, are summarized in Section 4. In Section 2, we present a general *cyclotomic Artin–Tate formula* for the special value of the Weil polynomial at a primitive mth root of unity and in Section 4 combine it with the sphere packing bounds to give a new class of *refined Artin–Tate* conditions generalizing Theorem 1.

As an application, showing that the new conditions are quite powerful, we record their effects on the list of candidate Weil polynomials of K3 surfaces over $\mathbb{F}_2$ computed by Kedlaya and Sutherland [25, Computation 3c].

Computation 1. *Of the* 1,672,565 *candidate Weil polynomials of K3 surfaces defined over* $\mathbb{F}_2$ *computed by Kedlaya and Sutherland:*

(a) *The quadratic refined Artin–Tate condition excludes* 628,290 *Weil polynomials.*

(b) *The cyclotomic refined Artin–Tate conditions (for* $m > 2$*) exclude* 658,298 *Weil polynomials.*

(c) *Together, all refined Artin–Tate conditions (for* $m \geq 2$*) exclude a total of* 878,580 *Weil polynomials, approximately* 52.53%.

After these conditions are applied, there remain 793,985 *candidate Weil polynomials of K3 surfaces defined over* $\mathbb{F}_2$.

There are two main ingredients in this work, both surrounding the discriminants of Frobenius coinvariant Néron–Severi lattices:

- upper bounds on these discriminants are extractable from the Weil polynomial by means of special values formulas at roots of unity (the cyclotomic Artin–Tate formulas);
- lower bounds on these discriminants follow from general sphere packing bounds on the minimum discriminants of even rootless positive definite lattices.

Combined, these yield new nontrivial lower bounds on special values of Weil polynomials of K3 surfaces over a finite field.

To establish the cyclotomic Artin–Tate formulas, we follow an idea of Elsenhans and Jahnel [16] to compare the Artin–Tate formula for a surface X over a finite field $\mathbb{F}_q$ with its base changes to finite extensions $\mathbb{F}_{q^m}$. In [16], the cyclotomic Artin–Tate formulas are obtained when the Picard rank of X does not jump after base change to $\mathbb{F}_{q_m}$, and more work is involved in establishing the formulas and making them useful in the case when the Picard rank does jump. Inspired by work of Geisser [19] and the Smith theory of coinvariant lattices [7, 6], we use calculations in Galois cohomology and the theory of gluing groups for coinvariant lattices to recast the cyclotomic Artin–Tate formula in terms of discriminants of Frobenius coinvariant lattices. Inspired by work of Bayer–Fluckiger [4, 5] on characteristic polynomials of finite order automorphisms of unimodular lattices, we can reinterpret the special values, which are actually norms from cyclotomic fields, in terms of resultants between cyclotomic polynomials and Weil polynomials, making the cyclotomic Artin–Tate formulas amenable to computation. Finally, we recall a well-known and important fact going back to Nikulin [36, 37] and Kondō [28], that coinvariant Néron–Severi lattices of a K3 surface (after changing their sign) are positive definite with no roots. For lattice of rank $k \leq 21$, we then use known values [13] (for $k \leq 8$) and bounds [11, 10] (for $k > 8$) on Hermite constants to give lower bounds on the discrimianants of even positive definite lattices with no roots. This yields nontrivial lower bounds on the discriminant of Frobenius coinvariant lattices, which we can turn into lower bounds on special values of Weil polynomials.

The article is organized as follows. Section 1 contains background on the arithmetic of the coinvariant lattice associated to a finite group of isometries. We review important consequences

of the Smith theory of the coinvariant lattice [7, 6] associated to a prime order group of isometries, and prove a generalization (Lemma 1.6) to all cyclic groups of isometries. Section 2 contains proofs of the quadratic and cyclotomic Artin–Tate formulas (Theorems 2.2 and 2.4). Section 3 records the fact, going back to Nikulin and Kondō, that coinvariant Néron–Severi lattices of a K3 surface (after changing their sign) are positive definite with no roots, and then reviews existing bounds in the sphere packing literature related to the Hermite constant and minimum discriminant of an even positive definite lattice with no roots. Section 4 ties together these ingredients.

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AI Disclosure. This project emerged from data (of Weil polynomials of smooth sextic and octic K3 surfaces over $\mathbb{F}_2$) that was generated in the second author's undergraduate thesis [40]. This data was processed using OpenAI GPT-6 Astra to search for statistical patterns beyond those imposed by the Weil conjectures and point count nonnegativity and growth conditions (as in [25]). Among other patterns, the model empirically discovered the quadratic refined Artin–Tate condition (and the connection with known sphere packing bounds) and suggested a proof strategy. Further prompting using OpenAI GPT-5.6, GPT-6, GPT-6 Astra, GPT-6 Sol, and Claude Opus 5 by the authors helped to uncover the general picture of the cyclotomic Artin–Tate formula and its related Weil polynomial conditions. We view this work as a further indication of how large datasets of mathematical objects can lead to the discovery of surprising connections, especially as AI/ML make data analysis work much more accessible. All text was written by the authors, who are responsible for its correctness.

1. Invariant and coinvariant sublattices

The study of the coinvariant lattive of a finite group of orthogonal transformations acting on a lattice has a long history going back to work of Nikulin [38, 36, 37], Mukai [33], and Kondō [27, 28], with applications to automorphisms on K3 surfaces. Various aspects of this theory were developed in [18, 20]. The Smith theory of a lattice as a module over the ring of cyclotomic integers was developed by Boissière, Nieper-Wißkirchen, and Sarti [7], which was used to study automorphisms of hyperkähler manifolds, a thread that was refined in [6, 8, 34]. We will need to generalize some aspects of the existing theory for our applications.

Let L be a lattice and G a finite subgroup of isometries. We denote by L^G the *invariant sublattice* and by $L_G = (L^G)^\perp$ the *coinvariant sublattice*. Denote by $N = \sum_{g\in G} g$ the norm operator on L. We recall the following, which is well-known.

Lemma 1.1. *Let G be a finite group of isometries of a lattice L. We have the following:*

(a) *L^G is orthogonal to L_G and $L^G \oplus L_G \subset L$ is of finite index,*
(b) *$L^G \subset L$ and $L_G \subset L$ are primitive sublattices,*
(c) *$L_G = \ker N$,*
(d) *$L/(L^G \oplus L_G)$ is killed by $|G|$.*
(e) *$L/(L^G \oplus L_G) \to L^G/|G|L^G$ defined by $v \mapsto N(v) \bmod |G|$ is injective.*

Proof. For (a), let $V = L\otimes\mathbb{Q}$, then $\frac{1}{|G|}N : V \to V$ is the projection onto the invariant subspace $V^G = L^G \otimes \mathbb{Q}$. Since G acts by isometries on L, we have that $\frac{1}{|G|}N$ is self-adjoint, and thus $\ker \frac{1}{|G|}N = (V^G)^\perp = V_G$. Thus we have an orthogonal decomposition $V = V^G \oplus V_G$. For (b), the fact that $L^G = L \cap V^G$ follows since L is torsion-free. The fact that $L_G = L \cap V_G$ follows from the fact that V_G is the orthogonal complement of V^G. For (c), we then have $\ker(N : L \to L) = L \cap V_G = L_G$.

For (d), we remark that $|G|v - N(v) = \sum_{g\in G} v - gv$ is contained in L_G for every $v \in L$; indeed, it is in the kernel of N, using the fact that $N^2 = |G|N$. Thus for every $v \in L$, we can write $|G|v = N(v) + (|G|v - N(v)) \in L^G \oplus L_G$, so the finite group $L/(L^G \oplus L_G)$ is killed by $|G|$, cf. [32, Lemma 1.4].

For (e), we first note that the map is well-defined since N restricts to multiplication by $|G|$ on L^G and to zero on L_G. Now assume that $N(v) = |G|w$ for some $w \in L^G$. Then $N(v - w) = 0$, which implies that $v - w \in L_G$, so that $v = w + (v - w) \in L^G \oplus L_G$. Thus the map is injective. □

The group $L/(L^G \oplus L_G)$ is often referred to as a *gluing group*, cf. [13, Chapter 4.3].

Remark 1.2. When G is a cyclic group of order m generated by σ, then $L \otimes \mathbb{C}$ decomposes into eigenspaces corresponding to eigenvalues that are mth roots of unity. Then $L^G \otimes \mathbb{C}$ is the $(+1)$-eigenspace and $L_G \otimes \mathbb{C}$ is the direct sum of all other eigenspaces.

In particular, when G is a cyclic group of order 2, generated by an involution σ, then $L^G = L_+$ is the $(+1)$-eigenlattice and $L_G = L_-$ is the (-1)-eigenlattice of σ.

When G is a cyclic group of order m generated by σ. Since $N(1 - \sigma) = 1 - \sigma^m = 0$, we have that $1 - \sigma : L \to L$ has kernel L^G and image contained in $L_G = \ker N$. Via restriction, we can then consider the endomorphism $1 - \sigma : L_G \to L_G$,

Lemma 1.3. *Let $G = \langle\sigma\rangle$ be a cyclic group of order $m > 1$ of isometries of a lattice L, and denote by $A = L/(L^G \oplus L_G)$. Then $1 - \sigma : L_G \to L_G$ has full rank and we have*

$$[L_G : (1 - \sigma)L_G] = \det\big(1 - \sigma \,|\, L_G \otimes \mathbb{Q}\big)$$

and

$$|H^1(G, L)| = [L_G : (1 - \sigma)L] = \frac{1}{|A|} \det\big(1 - \sigma \,|\, L_G \otimes \mathbb{Q}\big).$$

Proof. The fact that $1 - \sigma : L_G \to L_G$ has full rank (for $m > 1$) follows from Lemma 1.1. In particular, $1 - \sigma$ is injective when restricted to L_G. Since L_G is a free abelian group, we get the first equality, since we can compute the determinant of $1 - \sigma$ on a $\mathbb{Z}$-basis of L_G or on a $\mathbb{Q}$-basis of $L_G \otimes \mathbb{Q}$.

For the second equality, we have $H^1(G, L) \cong L_G/(1-F)L$ by the cohomology of cyclic groups. To compare the index of $(1-\sigma)L$ and of $(1-\sigma)L_G$ in L_G, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_G & \xrightarrow{1-\sigma} & L_G & \longrightarrow & L_G/(1-\sigma)L_G & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & L/L^G & \xrightarrow{1-\sigma} & L_G & \longrightarrow & L_G/(1-\sigma)L & \longrightarrow & 0 \end{array}$$

which implies by the snake lemma that the kernel of the right vertical map is isomorphic to $(L/L^G)/L_G \cong L/(L^G \oplus L_G)$ since $L^G \cap L_G = 0$. Thus $[L_G : (1 - F)L_G] = |A|[L_G : (1 - F)L]$. This proved the second equality. □

Remark 1.4. When σ is an involution on a lattice L with eigensublattices L^+ and L^-, then by Lemma 1.3 (and an immediate generalization), we have that $[L : L_+ \oplus L_-]\,[L_\pm : (1 \pm \sigma)L] = 2^{r_\pm}$, and in particular, that the $\mathbb{F}_2$-dimension of $L/(L_+ \oplus L_-)$ is bounded above by $\min(r_+, r_-)$. This fact is contained in Nikulin's work on involutions on lattices and 2-elementary lattices.

More generally, when G has prime order, one has the following restrictions on the coinvariant lattice and the "gluing group" $L/(L^G \oplus L_G)$, cf. [32, Lemma 1.8].

Lemma 1.5. *Let G be a group of prime order p acting on a lattice L. Then there exists $a_p \geq 0$ such that*

$$\operatorname{rk} L_G = a_p(p - 1).$$

Writing $[L : L^G \oplus L_G] = p^a$, we have

$$a \leq \min\{v_p(|L^G|), v_p(|L_G|), a_p\}$$

where $|L^G|$ and $|L_G|$ denote the absolute value of the discriminants of the invariant and coinvariant sublattices.

We provide a generalization of Lemma 1.5 to arbitrary cyclic group actions.

Lemma 1.6. *Let G be a cyclic group of order m acting on a lattice L. Then for each $d \mid m$, there exists $a_d \geq 0$, such that*

$$\operatorname{rk} L_G = \sum_{d|m,\, d>1} a_d \varphi(d).$$

Writing $A = L/(L^G \oplus L_G)$, then for every prime $p \mid m$, we have

$$v_p(|A|) \leq \min\{v_p(|L^G|), v_p(|L_G|), b_p(L_G)\}$$

where $|L^G|$ and $|L_G|$ denote the absolute value of the discriminants of the invariant and coinvariant sublattices, and where we define

$$b_p(L_G) = \sum_{i=1}^{v_p(m)} a_{p^i}$$

Also, we have that

$$\ell(A_p) \leq (\operatorname{rk} L_G)/(p-1)$$

where $\ell(A_p)$ denotes the number of cyclic factors of the p-primary part of A.

Proof. The action of $G = \langle \sigma \rangle$ extends to a representation of G on $V_G = L_G \otimes \mathbb{Q}$, yielding a decomposition

$$V_G \cong \bigoplus_{d|m,\, d>1} \rho_d^{\oplus a_d}$$

into nontrivial (since L_G is orthogonal to L^G) irreducible representations ρ_d of G, for some multiplicities $a_d \geq 0$. Here, $\rho_d = \mathbb{Q}(\zeta_d)$ with σ acting by multiplication by ζ_d. By Lemma 1.3, we have that

$$|H^1(G, L)| = [L_G : (1-\sigma)L] = \frac{1}{|A|} \det\big(1 - \sigma \,|\, V_G\big)$$

which, since $|H^1(G, L)|$ is an integer, shows that $|A|$ divides $\det(1 - \sigma \,|\, V_G)$. Since σ acts by multiplication by ζ_d on each summand ρ_d of V_G, the characteristic polynomial of σ on V_G is $\prod_{d|m,d>1} \Phi_d(T)^{a_d}$. Thus

$$\det\big(1 - \sigma \,|\, V_G\big) = \prod_{d|m,\, d>1} \Phi_d(1)^{a_d}.$$

However, by a formula first discovered by E. Lehmer [29, Theorem 3] and later in [1, 14], cf. [4, Lemma 4.1], we have

$$\Phi_d(1) = \begin{cases} 0 & \text{if } d = 1 \\ p & \text{if } d = p^k \text{ is a prime power with } k > 0 \\ 1 & \text{if } d \text{ is not a prime power} \end{cases} \tag{1.1}$$

which for $d > 1$ coincides with $e^{\Lambda(d)}$ where Λ is the von Mangoldt function from analytic number theory. Thus we have that

$$\det(1 - \sigma \,|\, V_G) = \prod_{p|m} p^{b_p(L_G)} \tag{1.2}$$

and so we arrive at the desired inequality of p-adic valuations

$$v_p(|A|) \leq b_p(L_G)$$

Since L^G and L_G are primitive in L, we have that $A = L/(L^G \oplus L_G)$ embeds into the direct sum $A_{L^G} \oplus A_{L_G}$ of discriminant groups, and projecting to each factor yields embeddings $A \subset A_{L^G}$ and $A \subset A_{L_G}$, see [38, Section 1.5]. This gives

$$v_p(|A|) \leq \min\{v_p(|L^G|), v_p(|L_G|)\},$$

which we combine with the previous inequality. Finally, to bound the number of cyclic factors of the p-primary part of A, we have that

$$\ell(A_p) \leq b_p(L_G) \leq \frac{1}{p-1} \sum_{i=1}^{v_p(m)} a_{p^i} \varphi(p^i) \leq \frac{\operatorname{rk} L_G}{p-1}$$

as desired. □

2. Artin–Tate special value formulas at roots of unity

Let X be a K3 surface over a finite field $\mathbb{F}_q$. We write $\rho = \operatorname{rk} \mathrm{NS}(X)$ for the Picard rank of X and $|\mathrm{NS}(X)|$ for the absolute value of the discriminant of the Néron–Severi lattice of X. We write $P_X(t) = q \det(1 - tF \mid H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)))$ for the normalized L-polynomial of Frobenius acting on ℓ-adic cohomology, which we call the *Weil polynomial* of X. Then $P_X(t) \in \mathbb{Z}[t]$ has degree 22, all roots of absolute value 1, and $P_X(0) = q$.

For a rational function $P(t) \in \overline{\mathbb{Q}}(t)$ and $a \in \overline{\mathbb{Q}}$ nonzero, we define the *special value* of P at a to be

$$P^*(a) = \lim_{t \to a} \frac{P(t)}{(1 - t/a)^r} = \lim_{t \to 1} \frac{P(at)}{(1-t)^r} \tag{2.1}$$

where r is the order of vanishing of $P(t)$ at $t = a$.

We recall the Artin–Tate formula, which is a special value formula for the Weil polynomial at $t = 1$,

$$P_X^*(1) = \lim_{t \to 1} \frac{P_X(t)}{(1-t)^\rho} = |\operatorname{Br}(X)|\,|\mathrm{NS}(X)| \tag{2.2}$$

whose validity has been established by the proof of the Tate conjecture for K3 surfaces over a finite field [39, 9, 30, 26, 31, 24]. We will be interested in comparing the Artin–Tate formula for various base changes of X.

To that end, for any $m \geq 1$ and any variety X over $\mathbb{F}_q$, we write $X_m = X_{\mathbb{F}_{q^m}}$ for the base change of X to $\mathbb{F}_{q^m}$ and $\rho_m = \operatorname{rk} \mathrm{NS}(X_m)$. We denote by $\mathrm{NS}_m(X) = \mathrm{NS}(X)^\perp \cap \mathrm{NS}(X_m)$ the coinvariant lattice of the (cyclic subgroup generated by the) Frobenius of X acting on $\mathrm{NS}(X_m)$, and by $A_m(X) = \mathrm{NS}(X_m)/(\mathrm{NS}(X) \oplus \mathrm{NS}_m(X))$ the corresponding gluing group. We also denote by $\operatorname{Br}_m(X) = \operatorname{coker}\big(\operatorname{Br}(X) \to \operatorname{Br}(X_m)\big)$. We take $\mathrm{NS}_1(X) = 0$, $A_1(X) = 0$, and $\operatorname{Br}_1(X) = 0$.

Since the action of Frobenius on $H^2_{\text{ét}}(\overline{X}_d, \mathbb{Q}_\ell(1))$ is the mth power of the action on $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$, the formula for the characteristic polynomial of the mth power of an operator immediately yields the following factorization formula for the Weil polynomial

$$P_{X_m}(t^m) = \prod_{\zeta^m = 1} P_X(\zeta t). \tag{2.3}$$

Here, we remind the reader that $P_{X_m} = q^m \det(1 - tF^m \mid H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)))$. We remark that, via the Weil conjectures, this factorization can also be derived from the generating function description of the zeta function, cf. [35, Proposition 2.14].

To state our special value formulas, we will need the following standard application of the Leray spectral sequence to the restriction map on the Brauer group, cf. [19, Section 2]. For the convenience of the reader, we provide a proof.

Lemma 2.1. *Let X be a smooth projective variety over a finite field $\mathbb{F}_q$. Then for every $m \geq 1$ we have*

$$\ker\big(\operatorname{Br}(X) \to \operatorname{Br}(X_m)\big) \cong H^1(G, \mathrm{NS}(X_m)).$$

where $G = \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$.

Proof. More generally, for any geometrically integral scheme X proper over a field k and a finite G-Galois extension field l/k, the Leray spectral sequence associated to the structure morphism $X_l \to X$ and the multiplicative group $\mathbb{G}_m$ (i.e., the Hochschild–Serre spectral sequence)

$$H^i(G, H^j(X_l, \mathbb{G}_m)) \Longrightarrow H^{i+j}(X, \mathbb{G}_m)$$

has sequence of low degree terms

$$\begin{aligned} 0 \longrightarrow H^1(G, l^\times) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_l)^G \longrightarrow H^2(G, l^\times) \\ \longrightarrow \ker\big(\mathrm{Br}(X) \to \mathrm{Br}(X_l)\big) \longrightarrow H^1(G, \mathrm{Pic}(X_l)) \longrightarrow H^3(G, l^\times). \end{aligned}$$

By Hilbert Theorem 90, we have that $H^1(G, l^\times) = 0$. If we now assume that l/k is a finite cyclic extension, then periodicity for cohomology of a cyclic group gives $H^3(G, l^\times) = H^1(G, l^\times) = 0$ and $H^2(G, l^\times) = k^\times / N_{l/k} L^\times$. Finally, when $k = \mathbb{F}_m$ is a finite field and $l = \mathbb{F}_{q^m}$, this later group is trivial. In this case, we deduce that the map

$$\ker\big(\mathrm{Br}(X) \to \mathrm{Br}(X_m)\big) \longrightarrow H^1(G, \mathrm{Pic}(X_m))$$

is an isomorphism. Now consider the short exact sequence

$$0 \to \mathrm{Pic}^0(X_m) \to \mathrm{Pic}(X_m) \to \mathrm{NS}(X_m) \to 0$$

of G-modules. Lang's theorem on vanishing of higher Galois cohomology of abelian varieties over a finite field gives $H^i(G, \mathrm{Pic}^0(X_m)) = 0$ for all $i \geq 1$, and then the long exact sequence yields an isomorphism $H^i(G, \mathrm{Pic}(X_m)) \to H^i(G, \mathrm{NS}(X_m))$ for all $i \geq 1$. □

As a warm-up for our special value calculations, we first give a formula, which is implicit in [16, Theorem 12], for the special value of the Weil polynomial of a K3 surface at -1. To that end, we remark that when $m = 2$, we have that $\mathrm{NS}_2(X) = \mathrm{NS}_-(X)$ is the (-1)-eigenlattice of Frobenius acting on $\mathrm{NS}(X_2)$, as in Remark 1.2. In analogy, we also write $\mathrm{Br}_-(X) = \mathrm{Br}_2(X)$.

Theorem 2.2. *Let X be a K3 surface over $\mathbb{F}_q$ with Weil polynomial $P_X(t)$. Then we have the following quadratic Artin–Tate formula*

$$P_X^*(-1) = 2^{\rho_2} \frac{|\mathrm{Br}(X_2)|}{|\mathrm{Br}(X)|} \frac{|\mathrm{NS}(X_2)|}{|\mathrm{NS}(X)|} = 2^{\rho} \frac{|\mathrm{Br}_-(X)|\,|\mathrm{NS}_-(X)|}{|A_2(X)|}.$$

Proof. By (2.3), we have $P_{X_2}(t^2) = P_X(t)\, P_X(-t)$. Dividing both sides by $(1-t^2)^{\rho_2}$, we have

$$\frac{P_{X_2}(t^2)}{(1-t^2)^{\rho_2}} = (1+t)^{-\rho_2} \frac{P_X(t)}{(1-t)^{\rho}} \frac{P_X(-t)}{(1-t)^{\rho^-}}$$

where we write $\rho_2 = \rho + \rho_-$, where $\rho_- = \mathrm{rk}\,\mathrm{NS}_-(X)$. Taking limits as $t \to 1$, we have

$$|\mathrm{Br}(X_2)|\,|\mathrm{NS}(X_2)| = 2^{-\rho_2}\,|\mathrm{Br}(X)|\,|\mathrm{NS}(X)|\,P_X^*(-1)$$

Where the left hand side follows from the Artin–Tate formula (2.2) for X_2, and the right hand side follows from the Artin–Tate formula for X and (by a change of variables $t \mapsto -t$) the special value at $t = -1$, which yields the first desired formula for $P_X^*(-1)$.

For the second formula, the exact sequence

$$0 \to \ker\big(\mathrm{Br}(X) \to \mathrm{Br}(X_2)\big) \to \mathrm{Br}(X) \to \mathrm{Br}(X_2) \to \mathrm{Br}_-(X) \to 0$$

combined with Lemmas 1.3 and 2.1, yields

$$\frac{|\mathrm{Br}(X_2)|}{|\mathrm{Br}(X)|} = \frac{|\mathrm{Br}_-(X)|}{|H^1(G, \mathrm{NS}(X_2))|} = \frac{|\mathrm{Br}_-(X)||A_2(X)|}{2^{\rho^-}}$$

Finally, by comparing discriminants, we have

$$|\mathrm{NS}(X)|\,|\mathrm{NS}_-(X)| = |\mathrm{NS}(X) \oplus \mathrm{NS}_-(X)| = |A_2(X)|^2\,|\mathrm{NS}(X_2)|$$

from which the stated result follows. □

As a corollary, we arrive at (a refinement of) the condition of Elsenhans and Jahnel [16].

Corollary 2.3. *Let X be a K3 surface over $\mathbb{F}_q$ with Weil polynomial $P_X(t)$. Assume that $\rho = \rho_2$. Then $\mathrm{NS}(X) \to \mathrm{NS}(X_2)$ is an isomorphism and $\mathrm{Br}(X) \to \mathrm{Br}(X_2)$ is injective, hence the quadratic Artin–Tate formula reduces to*

$$P_X^*(-1) = 2^\rho \, |\mathrm{Br}_-(X)|.$$

where $\mathrm{Br}_-(X)$ has (integral) square order.

Proof. The base change map $\mathrm{NS}(X) \to \mathrm{NS}(X_2)$ is injective by a restriction-corestriction argument. Hence the (-1)-eigenspace of Frobenius acting on $\mathrm{NS}(X_2) \otimes \mathbb{Q}$ is trivial. And since $\mathrm{NS}(X_2)$ is torsion-free, the (-1)-eigenlattice $\mathrm{NS}_-(X)$ must be trivial as well. Now, for $v \in \mathrm{NS}(X_2)$ we have that $F(v) - v$ is contained in $\mathrm{NS}_-(X)$, hence must be zero, showing that $v \in \mathrm{NS}(X_2)^F = \mathrm{NS}(X)$, where this last equality follows from the Leray spectral sequence associated $X_2 \to X$ and the sheaf $\mathbb{G}_m$, the triviality of the Picard scheme of a K3 surface, and the fact that the Brauer group of a finite field is trivial. Thus the base change map $\mathrm{NS}(X) \to \mathrm{NS}(X_2)$ is surjective as well.

By Lemma 2.1, the kernel of the base change map $\mathrm{Br}(X) \to \mathrm{Br}(X_2)$ is isomorphic to $H^1(G, \mathrm{NS}(X_2))$, where G is generated by Frobenius. This later group is trivial as $\mathrm{NS}(X_2)$ is a free abelian group with trivial G-action. Hence the base change map on Brauer groups is injective, and the cokernel $\mathrm{Br}_-(X)$ has order $|\mathrm{Br}(X_2)|/|\mathrm{Br}(X)|$, which is an integer that is a quotient of squares (as the Brauer group of a surface over a finite field is a square), hence is an integer square. □

Now, we present the following generalization of Theorem 2.2 to give an Artin–Tate formula for special values at higher degree roots of unity. To that end, for a variety X over $\mathbb{F}_q$ define $b_1(X) = 1$ and for $m \geq 2$ define

$$b_m(X) = \prod_{p \mid m} p^{b_p(\mathrm{NS}_m(X))}$$

where the product is taken over primes p dividing m and where the invariant b_p of a coinvariant lattice is introduced in the statement of Lemma 1.6. For example, when $m = p$ is prime Lemma 1.5 gives $b_p = p^{\mathrm{rk}\,\mathrm{NS}_p(X)/(p-1)}$. In particular, in the context of Theorem 2.2, $b_2 = 2^{\rho-}$.

Theorem 2.4. *Let X be a K3 surface over $\mathbb{F}_q$ with Weil polynomial $P_X(t)$. Then for every $m \geq 2$ we have the following cyclotomic Artin–Tate formula*

$$\begin{aligned} N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_X^*(\zeta_m)) &= \prod_{d \mid m} \left(d^{\rho_d} \, |\mathrm{Br}(X_d)| \, |\mathrm{NS}(X_d)|\right)^{\mu(m/d)} \\ &= \prod_{d \mid m} \left(\frac{d^{\rho_d} \, |\mathrm{Br}_d(X)| \, |\mathrm{NS}_d(X)|}{|A_d(X)| \, b_d(X)} \right)^{\mu(m/d)}. \end{aligned}$$

Proof. The proof is very similar to the proof of Theorem 2.2, but a bit more elaborate in parts. By (2.3), we have $P_{X_m}(t^m) = \prod_{\zeta^m=1} P_X(\zeta t)$. Writing ρ_ζ for the algebraic multiplicity of ζ as an eigenvalue of Frobenius acting on $\mathrm{NS}(X_m)$, we have that $\rho_m = \sum_{\zeta^m=1} \rho_\zeta$. Dividing both sides by $(1-t^m)^{\rho_m}$, we have

$$\frac{P_{X_m}(t^m)}{(1-t^m)^{\rho_m}} = \frac{1}{(1+t+\cdots+t^{m-1})^{\rho_m}} \prod_{\zeta^m=1} \frac{P_X(\zeta t)}{(1-t)^{\rho_\zeta}}$$

and taking limits as $t \to 1$ and using (2.1), we have

$$m^{\rho_m} \, P_{X_m}^*(1) = \prod_{\zeta^m=1} P_X^*(\zeta) = \prod_{d \mid m} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_X^*(\zeta_d))$$

where we use the fact that $P_X(t)$ has rational coefficients (in fact, integral coefficients, but we don't need that here). Möbius inversion and the Artin–Tate formula (2.2) for X_m then give

$$N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_X^*(\zeta_m)) = \prod_{d|m} \bigl(d^{\rho_d} \,|\operatorname{Br}(X_d)|\,|\operatorname{NS}(X_d)|\bigr)^{\mu(m/d)}$$

which is the first claimed equality in the formula.

For the second equality, the exact sequence

$$0 \to \ker\bigl(\operatorname{Br}(X) \to \operatorname{Br}(X_d)\bigr) \to \operatorname{Br}(X) \to \operatorname{Br}(X_d) \to \operatorname{Br}_d(X) \to 0$$

combined with Lemmas 1.3 and 2.1 yields

$$|\operatorname{Br}(X_d)| = |\operatorname{Br}(X)| \, \frac{|\operatorname{Br}_d(X)|}{|H^1(G, \operatorname{NS}(X_d))|}.$$

By comparing discriminants, we have

$$|\operatorname{NS}(X)|\,|\operatorname{NS}_d(X)| = |\operatorname{NS}(X) \oplus \operatorname{NS}_d(X)| = |A_d(X)|^2\,|\operatorname{NS}(X_d)|$$

Thus, using Lemmas 1.3 and 2.1, we have

$$|\operatorname{Br}(X_d)|\,|\operatorname{NS}(X_d)| = |\operatorname{Br}(X)|\,|\operatorname{NS}(X)| \, \frac{|\operatorname{Br}_d(X)|\,|\operatorname{NS}_d(X)|}{b_d\,|A_d(X)|}$$

where we temporarily set $b_d = |\det(1 - F \mid \operatorname{NS}_d(X) \otimes \mathbb{Q})|$. By (1.2), we compute that

$$b_d = \det(1 - F \mid \operatorname{NS}_d(X) \otimes \mathbb{Q}) = \prod_{p|m} p^{b_p(\operatorname{NS}_d(X))} = b_d(X).$$

Finally, since $\prod_{d|m} c^{\mu(m/d)} = 1$ for any $c \neq 0$ and any $m \geq 2$, we have that

$$N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_X^*(\zeta_m)) = \prod_{d|m} \left(\frac{d^{\rho_d}\,|\operatorname{Br}_d(X)|\,|\operatorname{NS}_d(X)|}{b_d(X)\,|A_d(X)|} \right)^{\mu(m/d)}$$

which is the second stated equality. □

Remark 2.5. Considering a different normalization of the special value leads to some simplifications. To this end, write

$$P_X(t) = (1-t)^{\rho}\, P_1(t).$$

Then $P_X^*(\zeta_m) = (1-\zeta_m)^{\rho} P_1^*(\zeta_m)$ implies that

$$N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_X^*(\zeta_m)) = \Phi_m(1)^{\rho} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_1^*(\zeta_d))$$

By (1.1), we have $\prod_{d|m,\, d>1} \Phi_d(1) = m$, which implies that $\prod_{d|m} d^{\mu(m/d)} = \Phi_m(1)$ for $m \geq 2$ by Möbius inversion. Hence Theorem 2.4 can be restated as

$$\begin{aligned} N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_1^*(\zeta_m)) &= \prod_{d|m} \bigl(d^{\rho_d - \rho}\,|\operatorname{Br}(X_d)|\,|\operatorname{NS}(X_d)|\bigr)^{\mu(m/d)} \\ &= \prod_{d|m} \left(\frac{d^{\rho_d - \rho}\,|\operatorname{Br}_d(X)|\,|\operatorname{NS}_d(X)|}{|A_d|\,b_d(X)} \right)^{\mu(m/d)} \end{aligned} \tag{2.4}$$

In the setting of Corollary 2.3, when $\rho = \rho_2$ we know that $(1+t)$ does not divide $P_X(t)$, and we have the strikingly simple formula

$$P_1(-1) = P_1^*(-1) = |\operatorname{Br}^-(X)|$$

for the order of the growth of the Brauer group under base change to $\mathbb{F}_{q^2}$.

We have the following generalization of the construction of Elsenhans and Jahnel [16] to the higher cyclotomic case, which is analogous to (and proved in the same way as) Corollary 2.3.

Corollary 2.6. *Let X be a K3 surface over $\mathbb{F}_q$ with Weil polynomial $P_X(t)$. For some $m \geq 2$, assume that $\rho = \rho_m$. Then $\mathrm{NS}(X) \to \mathrm{NS}(X_m)$ is an isomorphism and $\mathrm{Br}(X) \to \mathrm{Br}(X_m)$ is injective, and the cyclotomic Artin–Tate formula reduces to*

$$N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_X^*(\zeta_m)) = \Phi_m(1)^\rho \prod_{d|m} |\mathrm{Br}_d(X)|^{\mu(m/d)}$$

or equivalently

$$N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_1^*(\zeta_m)) = \prod_{d|m} |\mathrm{Br}_d(X)|^{\mu(m/d)}$$

where the alternative product of Brauer cokernels is an integral square.

Proof. The statements about the base change maps on the Néron–Severi and Brauer groups follow exactly as in the proof of Corollary 2.3. In this case, $\rho = \rho_d$ and $|\mathrm{NS}_d(X)| = |A_d(X)| = b_d = 1$ for all $d \mid m$, hence we arrive at the desired formulas by applying (1.1) and Remark 2.5.

Finally, we consider the alternative product of Brauer cokernels. Since the order of the Brauer group of a K3 surface over a finite field is a square and $\mathrm{Br}(X) \to \mathrm{Br}(X_d)$ is injective for all $d \mid m$, we have that $\mathrm{Br}_d(X)$ has square order. Thus the alternative product is a rational square. Since $P_1(t)$ has integer coefficients, $N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_1^*(\zeta_m))$ is an algebraic integer, and thus so is the alternating product of Brauer cokernels. A rational square that is an algebraic integer is an integer square, yielding the final claim. □

Remark 2.7. We now make a comparison with the construction of [16, Section 5]. Elsenhans and Jahnel define an invariant $E^{(m)}$ associated to any degree 22 Weil polynomial with roots of absolute value q. Translating to our Weil polynomial convention (roots of absolute value 1), we have that

$$E^{(m)} = q^{m-1} \prod_{\alpha \neq 1} \frac{1-\alpha^m}{1-\alpha}$$

where the product is over all inverse roots α of a Weil polynomial P. When $P = P_X$ for a K3 surface X over $\mathbb{F}_q$, we see that $E^{(m)} = 0$ if and only if $\rho_m > \rho$, cf. [16, Remark 13]. In the case $\rho = \rho_m$, we have that

$$E^{(m)} = \prod_{\zeta^m=1,\, \zeta \neq 1} P_1(\zeta) = \prod_{d|m,\, d>1} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_1(\zeta)) = \frac{1}{m^\rho} \prod_{d|m,\, d>1} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_X^*(\zeta_d)).$$

The first equation follows from the fact that $(1-\alpha^m)/(1-\alpha) = \prod_{\zeta^m=1,\, \zeta\neq 1}(1-\zeta\alpha)$. The final equation follows from Remark 2.5. On the other hand, still assuming $\rho = \rho_m$, we have

$$E^{(m)} = \frac{P_{X_m}^*(1)}{P_X^*(1)} = \frac{|\mathrm{Br}(X_m)|\,|\mathrm{NS}(X_m)|}{|\mathrm{Br}(X)|\,|\mathrm{NS}(X)|} = |\mathrm{Br}_m(X)|$$

using the classical Artin–Tate formula and the fact that $|\mathrm{Br}_m(X)| = |\mathrm{Br}(X_m)|/|\mathrm{Br}(X)|$. This gives a refinement of the fact that $E^{(m)}$ is a square, see [16, Theorem 12].

3. Sphere packing bounds

In this section, we survey the existing literature for lower bounds on the discriminants of lattices arising in the theory of sphere-packing. Such bounds are useful for K3 surfaces over a finite field because our cyclotomic Artin–Tate formulas give upper bounds for the discriminants of Frobenius coinvariant lattices in terms of special values of the Weil polynomial. The following result, which in spirit goes back to Nikulin [36, 37], cf. Kondō [28, Proposition 2] motivates the entire approach.

Lemma 3.1. *Let X be a smooth K3 surface over a field k, and let l/k be an extension. Then the sublattice $N = \mathrm{NS}(X)^\perp \cap \mathrm{NS}(X_l)$ is even, negative definite, and has no (-2)-classes.*

Proof. Let H be an ample divisor on X defined over k. It is a standard fact that $\mathrm{NS}(X_l)$ is even, hence so is any sublattice. Since $N \subset H^\perp$, the Hodge index theorem implies that N is negative definite. We now prove that this sublattice is rootless. In the case where $N = 0$ the statement is trivial. Assume that $C \in N$ satisfies $C^2 = -2$, then either C or $-C$ is effective (cf. [22, Section 2.1.5]), in which case $H.C = 0$ is not possible by Nakai–Moishezon–Kleiman (cf. [22, Theorem 8.1.2]). Hence N has no (-2)-classes. □

Note that the conclusion of Lemma 3.1 does not imply that X cannot acquire (-2)-classes over an extension, but just that any (-2)-class cannot be orthogonal to $\mathrm{NS}(X)$.

Remark 3.2. For l/k a finite G-Galois extension, note that, in the notation of Lemma 3.1, $N = N_G$ is the coinvariant lattice of the induced action of G on $\mathrm{NS}(X_l)$. Hence, in spirit, Lemma 3.1 is an analogue of Nikulin [37, Lemma 4.2].

Now, let $L = N(-1)$ be obtained by changing the sign of the intersection form. Then L is even, positive definite, and *rootless*, i.e., contains no vectors of norm 2. In particular, the norm of any vector in K is at least 4. Positive definite lattices satisfying this property (even or not) have been extensively studied as they describe integral lattice sphere packings.

More specifically, for a positive definite lattice L of rank k, define its *(Hermitian) minimum* $\mu(L)$ to be the least norm of a non-zero vector in L and and its *discriminant* $D(L)$ to be the determinant of a Gram matrix of an integral basis of L. Then the minimum distance between distinct lattice points is $\sqrt{\mu(L)}$ and the volume of a fundamental domain for L is given by $\sqrt{D(L)}$. Consequently, balls of radius $\frac{1}{2}\sqrt{\mu(L)}$ centered at the lattice points form a packing of density

$$\Delta(L) = \frac{v_k}{2^k}\frac{\mu(L)^{k/2}}{D(L)^{1/2}}$$

where v_k is the volume of the unit ball in $\mathbb{R}^k$.

The classical problem of finding the densest lattice sphere packing in dimension k is equivalent to determining the *Hermite constant*

$$\gamma_k = \sup_L \frac{\mu(L)}{D(L)^{1/k}},$$

where L ranges over all positive definite lattices of rank k. Indeed, let Δ_k be the largest density of a lattice sphere packing in dimension k. Then we have that

$$\Delta_k = \frac{v_k}{2^k}\gamma_k^{k/2}.$$

For our purposes, we can use Hermite's inequality $\mu(L) \leq \gamma_k D(L)^{1/k}$ to provide a lower bound on $D(L)$ in terms of the minimum $\mu(L)$. Let δ_k denote the least discriminant of a positive definite even rootless lattice of rank k, hence with minimum at least 4. Then we have

$$D(L) \geq \delta_k \geq \left(\frac{4}{\gamma_k}\right)^k.$$

The exact Hermite constants are known only in rank $k \leq 8$, see [13, Chapter 1], and $k = 24$ [12]. In rank up to 8, we have

k	1	2	3	4	5	6	7	8
γ_k^k	1	$\frac{4}{3}$	2	4	8	$\frac{64}{3}$	64	256
δ_k	4	12	32	64	128	192	256	256

which are attained by the scaled even root lattices $A_1(2)$, $A_2(2)$, $A_3(2)$, $D_4(2)$, $D_5(2)$, $E_6(2)$, $E_7(2)$, and $E_8(2)$ [13, Chapter 1]. In rank 24, the Hermite constant is attained by the Leech lattice. In the remaining ranks up to 21, we can appeal to numerical upper-bounds on the

Hermite constant established by Cohn and Elkies [11], and updated by Cohn, de Laat, and Salmon [10] to provide lower-bounds on lattice discriminants.

k	9	10	11	12	13	14	15	16	17	18	19	20	21
$\delta_k \geq$	298	317	312	281	230	177	128	89	58	37	23	13	8

Remark 3.3. By Lemma 3.1, the negatives of coinvariant Néron–Severi lattices of K3 surfaces are positive definite even rootless lattices, which is why we restrict our attention to even lattices. Since the Hermite constant is invariant under scaling, we can always assume that lattices are even for the purposes of bounding the Hermite constant. However, the minimum discriminant of positive definite even rootless lattices, can a priori be larger than the minimum discriminant of all positive definite lattices with minimum at least 4. Thus the invariant δ_k that we care about may satisfy even stronger bounds than those achievable via an analysis of the Hermite constant, see [2] for example. When $k \leq 8$ and $k = 24$, in fact the even lattices achieving the bound (listed above) have minimum 4. Whether it's true that the Hermite constant in any rank k is realized by a positive definite even lattice of minimum 4, is an open question.

4. Refined Artin–Tate conditions

In this section, we use the sphere packing bounds in Section 3, together with the quadratic and cyclotomic Artin–Tate formulas, to establish new nontrivial conditions on the possible Weil polynomials of K3 surfaces over a finite field. We call these *refined Artin–Tate conditions*.

Theorem 4.1. *Let X be a K3 surface over $\mathbb{F}_q$ with Weil polynomial $P_X(t)$. Fix $m \geq 2$ and write*

$$P_X(t) = (1-t)^\rho \, C_m(t) \, P_m(t)$$

where $C_m(t) = \prod_{d\mid m,\, d>1} \Phi_d(t)^{a_d}$ for $a_d \geq 0$, where $P_m(\zeta_d) \neq 0$ for all $d \mid m$, and $P_m(0) = q$. Then we have

$$|\operatorname{NS}_m(X)| = \frac{|A_m(X)|}{|\operatorname{Br}_m(X)|} \prod_{d\mid m,\, d>1} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_m(\zeta_d)).$$

Concretely, we have that

$$\operatorname{Res}\left(\frac{t^m-1}{t-1}, P_m(t)\right) = \frac{|\operatorname{Br}_m(X)|\,|\operatorname{NS}_m(X)|}{|A_m(X)|}$$

and when $\rho < \rho_m$, this yields the following refined Artin–Tate condition

$$\gcd\big(m^\rho, C_m(1)\big) \operatorname{Res}\left(\frac{t^m-1}{t-1}, P_m(t)\right) \geq \delta_{\rho_m - \rho}$$

where Res *is the classical resultant and δ_r is the minimum discriminant of an even positive definite rootless lattice of rank r.*

Proof. Applying Möbius inversion to the first version of Theorem 2.4 yields

$$\prod_{d\mid m} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_X^*(\zeta_d)) = m^{\rho_m}\,|\operatorname{Br}(X_m)|\,|\operatorname{NS}(X_m)|.$$

Now write $G_m(t) = (1-t)^\rho \, C_m(t)$, and notice that

$$\prod_{d\mid m} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(G_m^*(t)) = \prod_{\zeta^m=1} G_m^*(\zeta) = m^{\rho_m}$$

since all roots of G_m are mth roots of unity. Since $P_X^*(\zeta) = G_m^*(\zeta)\, P_m(\zeta)$ for any mth root of unity ζ, we obtain

$$\prod_{d\mid m} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_m(\zeta_d)) = |\operatorname{Br}(X_m)|\,|\operatorname{NS}(X_m)|$$

which can be viewed as a version of the Artin–Tate formula for $P_m(t)$. The classical Artin–Tate formula gives $P_X^*(1) = |\operatorname{Br}(X)|\,|\operatorname{NS}(X)| = C_m(1)\,P_m(1)$. Hence dividing both sides of the above by $P_m(1)$ yields

$$\prod_{d|m,\,d>1} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_m(\zeta_d)) = \frac{C_m(1)\,|\operatorname{Br}(X_m)|\,|\operatorname{NS}(X_m)|}{|\operatorname{Br}(X)|\,|\operatorname{NS}(X)|} = \frac{|\operatorname{Br}_m(X)|}{|A_m(X)|}|\operatorname{NS}_m(X)|$$

using the same identities as in the proof of Theorem 2.4. This is the desired formula.

For the concrete statement, we first remark that classical formulas for the resultant, cf. [1], yield

$$\operatorname{Res}(t^{m-1}+\cdots+1, P_m(t)) = \prod_{\zeta^m=1,\,\zeta\neq 1} P_m(\zeta_d) = \prod_{d|m,\,d>1} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_m(\zeta_d)).$$

Second, Lemma 1.6 implies that $|A_m(X)|$ divides $b_m(X) = C_m(1)$ and Lemma 1.1(e) implies that $|A_m(X)|$ divides m^ρ. Third, $|\operatorname{Br}_m(X)|$ is an integer. Thus we have that

$$\gcd(m^\rho, C_m(1))\operatorname{Res}(t^{m-1}+\cdots+1, P_m(t)) \geq |\operatorname{NS}_m(X)| \geq \delta_{\rho_m-\rho}$$

since, by Lemma 3.1, $\operatorname{NS}_m(X)$ is (up to sign) a positive definite even rootless lattice, which has rank $\rho_m - \rho$, hence by Section 3, its discriminant is bounded by $\delta_{\rho_m-\rho}$. □

The introduction of the resultant in the statement of Theorem 4.1 is only to provide a more computationally practical way of computing the $\prod_{d|m,\,d>1} N_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(P_m(\zeta_d))$ term. As we shall see, these refined Artin–Tate conditions can be effectively used to rule out whether a Weil polynomial $P(t)$ of degree 22 can be realized by a K3 surface over $\mathbb{F}_q$.

To sharpen the refined Artin–Tate conditions a bit, we can apply them to a Weil polynomial $P(t) = \prod_i(1-\alpha_i t)$ and to all its formal base changes $P^{(l)}(t) = \prod_i(1-\alpha_i^l t)$. In fact, there is only a finite number of possible base changes to check before the entire geometric Picard group is defined over $\mathbb{F}_{q^l}$, and the conditions (which require further Picard rank jumping after an extension) are no longer valid. Similarly, there is only a finite number of refined Artin–Tate conditions (a finite number of m) to check, governed by the possible factorizations of the algebraic part of the Weil polynomial into cyclotomic polynomials.

In the case $\rho = \rho_m$, the refined Artin–Tate condition doesn't apply. However, Corollary 2.3 gives a potential condition, namely that $N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(P_1^*(\zeta_m))$ is a square. Unfortunately, in view of Remark 2.7, a result of Elsenhans and Jahnel [16, Corollary 18] implies that this condition is automatically satisfied by any Weil polynomial when m is odd, and is satisfied for all m as soon as it is satisfied for $m = 2$, reducing us to Elsenhans and Jahnel's original condition.

We now review how the refined Artin–Tate conditions sort out the list of candidate Weil poluynomials for K3 surfaces over $\mathbb{F}_2$ computed by Kedlaya and Sutherland [25].

Let $\mathcal{K}$ be the set of 2,258,428 candidate Weil polynomials of degree 22 that satisfy only the point count nonnegativity and extension growth properties for K3 surfaces over $\mathbb{F}_2$, cf. [3, Theorem 3.4]. We consider the subsets $E_2, R_2, R_{>2}$ of $\mathcal{K}$ consisting of Weil polynomials that fail the original Elsenhans and Jahnel quadratic Artin–Tate condition (when $\rho = \rho_2$), fail the quadratic refined Artin–Tate condition (when $m = 2$), and fail the refined Artin–Tate condition for all $m > 2$, respectively. As discussed above, in computing these subsets, we also take into account evaluating each condition after performing formal base changes to $\mathbb{F}_{q^l}$. In particular, $\mathcal{K} \smallsetminus E_2$ is the set of 1,672,565 candidate Weil polynomials of K3 surfaces over $\mathbb{F}_2$ in [25, Computation 3c].

In Kedlaya and Sutherland's candidate list $\mathcal{K} \smallsetminus E_2$, the condition $R_{>2}$ excludes 250,290 candidates that survive every quadratic condition. Conversely, the quadratic conditions E_2, R_2 exclude 220,282 candidates that survive every higher degree cyclotomic condition. Together, the new conditions exclude 878,580 of the 1,672,565 Kedlaya–Sutherland candidates (52.53%), leaving a candidate list of 793,985 Weil polynomials.

Subset $S \subset \mathcal{K}$	Alone $\lvert S\rvert$	New beyond E_2 $\lvert S \setminus E_2\rvert$	With E_2 $\lvert S \cup E_2\rvert$
Original E-J quadratic (E_2)	585,863	0	585,863
Refined quadratic (R_2)	770,118	628,290	1,214,153
Refined cyclotomic ($R_{>2}$)	870,749	658,298	1,244,161
Total	1,464,443	878,580	1,464,443

TABLE 1. Exclusion counts for the three types of Artin–Tate conditions in $\mathcal{K}$.

REFERENCES

1. Tom M. Apostol, *Resultants of cyclotomic polynomials*, Proc. Amer. Math. Soc. **24** (1970), 457–462.
2. Asher Auel, Phil Engel, Kiran Kedlaya, and Sair Shaikh, *Pointless K3 surfaces over finite fields*, preprint, 2026.
3. Asher Auel and Jack Petok, *Zeta functions of K3 categories over finite fields*, Res. Number Theory **12** (2026), no. 96, 2026.
4. Eva Bayer-Fluckiger, *K3 surfaces, cyclotomic polynomials and orthogonal groups*, Eur. J. Math. **10** (2024), no. 2, Paper No. 36, 30.
5. ———, *Automorphisms of K3 surfaces, signatures, and isometries of lattices*, J. Eur. Math. Soc. (2025).
6. Samuel Boissière, Chiara Camere, Giovanni Mongardi, and Alessandra Sarti, *Isometries of ideal lattices and hyperkähler manifolds*, Int. Math. Res. Not. IMRN (2016), no. 4, 963–977.
7. Samuel Boissière, Marc Nieper-Wißkirchen, and Alessandra Sarti, *Smith theory and irreducible holomorphic symplectic manifolds*, J. Topol. **6** (2013), no. 2, 361–390.
8. Simon Brandhorst and Alberto Cattaneo, *Prime order isometries of unimodular lattices and automorphisms of ihs manifolds*, 2020.
9. François Charles, *The Tate conjecture for K3 surfaces over finite fields*, Invent. Math. **194** (2013), 119–145.
10. Henry Cohn, David de Laat, and Andrew Salmon, *Three-point bounds for sphere packing*, 2022, Preprint, arXiv:2206.15373.
11. Henry Cohn and Noam Elkies, *New upper bounds on sphere packings I*, Ann. of Math. (2) **157** (2003), no. 2, 689–714.
12. Henry Cohn and Abhinav Kumar, *Optimality and uniqueness of the Leech lattice among lattices*, Annals of Mathematics **170** (2009), no. 3, 1003–1050.
13. J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, third ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1999, With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.
14. Fritz-Erdmann Diederichsen, *Über die Ausreduktion ganzzahliger Gruppendarstellungen bei arithmetischer Äquivalenz*, Abh. Math. Sem. Hansischen Univ. **13** (1940), 357–412.
15. Andreas-Stephan Elsenhans, *Weil polynomials of K3-surfaces*, talk at Algorithmic Number Theory Symposium (ANTS IX), INRIA, Nancy, France.
16. Andreas-Stephan Elsenhans and Jörg Jahnel, *On Weil polynomials of K3 surfaces*, Algorithmic Number Theory, Lecture Notes in Computer Science, vol. 6197, Springer, Berlin, 2010, pp. 126–141.
17. Andreas-Stephan Elsenhans and Jörg Jahnel, *On the characteristic polynomial of the Frobenius on étale cohomology*, Duke Math. J. **164** (2015), no. 11, 2161–2184.
18. Alice Garbagnati and Alessandra Sarti, *Symplectic automorphisms of prime order on K3 surfaces*, J. Algebra **318** (2007), no. 1, 323–350.
19. Thomas H. Geisser, *Brauer and Néron–Severi groups of surfaces over finite fields*, J. Number Theory **284** (2026), 38–61.
20. Kenji Hashimoto, *Finite symplectic actions on the K3 lattice*, Nagoya Math. J. **206** (2012), 99–153.
21. Taira Honda, *Isogeny classes of abelian varieties over finite fields*, J. Math. Soc. Japan **20** (1968), 83–95.
22. Daniel Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016.
23. Kazuhiro Ito, *Unconditional construction of K3 surfaces over finite fields with given L-function in large characteristic*, Manuscripta Math. **159** (2019), no. 3-4, 281–300.
24. Kazuhiro Ito, Tetsushi Ito, and Teruhisa Koshikawa, *CM liftings of K3 surfaces over finite fields and their applications to the Tate conjecture*, Forum Math. Sigma **9** (2021), e29, 70 pp.
25. Kiran S. Kedlaya and Andrew V. Sutherland, *A census of zeta functions of quartic K3 surfaces over* $\mathbb{F}_2$, LMS J. Comput. Math. **19** (2016), no. A, 1–11, arXiv:1511.06945.

26. Wansu Kim and Keerthi Madapusi Pera, 2-*adic integral canonical models*, Forum Math. Sigma **4** (2016), e28, 34 pp.
27. Shigeyuki Kondō, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), no. 1, 75–98.
28. ———, *Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of K3 surfaces*, Duke Math. J. **92** (1998), no. 3, 593–603, With an appendix by Shigeru Mukai.
29. Emma T. Lehmer, *A numerical function applied to cyclotomy*, Bull. Amer. Math. Soc. **36** (1930), no. 4, 291–298.
30. Keerthi Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. **201** (2015), no. 2, 625–668.
31. ———, *Erratum to appendix to* 2-*adic integral canonical models*, Forum Math. Sigma **8** (2020), e14, 11 pp.
32. Giovanni Mongardi, Kévin Tari, and Malte Wandel, *Prime order automorphisms of generalised Kummer fourfolds*, Manuscripta Math. **155** (2018), no. 3–4, 449–469.
33. Shigeru Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group M_{23}*, Invent. Math. **94** (1988), no. 1, 183–221.
34. Stevell Muller, *Algebraically trivial automorphisms of irreducible holomorphic symplectic manifolds*, 2025.
35. Mircea Mustaţă, *Zeta functions in algebraic geometry*, Lecture notes available at https://public.websites.umich.edu/~mmustata/zeta_book.pdf, 2019.
36. Viacheslav V. Nikulin, *Finite groups of automorphisms of Kählerian surfaces of type K3*, Russ. Math. Surv. **31** (1976), no. 2, 247–248, English translation of *Uspekhi Mat. Nauk*, 31:2(188) (1976), 223–224.
37. ———, *Finite groups of automorphisms of Kählerian surfaces of type K3*, Trans. Mosc. Math. Soc. **38** (1980), no. 2, 71–135, English translation of *Tr. MMO*, 38 (1979), 75–137.
38. ———, *Integral symmetric bilinear forms and some of their applications*, Math. USSR-Izv. **14** (1980), no. 1, 103–167, English translation of *Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 111–177, 238.*
39. Niels O. Nygaard and Arthur Ogus, *Tate's conjecture for K3 surfaces of finite height*, Ann. of Math. (2) **122** (1985), 461–507.
40. Sair Shaikh, *Census of sextic and octic K3 surfaces over $\mathbb{F}_2$*, Undergraduate Thesis, Dartmouth College, 2026.
41. Lenny Taelman, *K3 surfaces over finite fields with given L-function*, Algebra Number Theory **10** (2016), no. 5, 1133–1146.
42. ———, *Ordinary K3 surfaces over a finite field*, J. Reine Angew. Math. **761** (2020), 141–161.
43. John Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

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