**Failure of the local-global principle for isotropy of quadratic forms over surfaces**

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Let $X$ be an integral scheme, $K$ its function field, $\Omega$ the set of rank 1 discrete valuations on $K$, and $K_v$ the completion of $K$ at $v \in \Omega$. We assume throughout that 2 is invertible on $X$. Let $q$ be a nondegenerate quadratic form over $K$ and $q_v = q \otimes_K K_v$. The *local-global principle* for isotropy of quadratic forms is the statement: if $q_v$ is isotropic over $K_v$ for all $v \in \Omega$ then $q$ is isotropic over $K$. A natural question is: does the local-global principle hold for a given function field $K$?

We mention three examples. First, the local-global principle holds if $K$ is a global field by the Hasse–Minkowski theorem. Second, let $K$ be the function field of a smooth proper curve $X$ over an algebraically closed field $k$. Here, $\Omega$ is in bijection with the set of closed points of $X$. By Tsen's theorem, all quadratic forms of dimension $\geq 3$ are isotropic. An anisotropic form $q$ of dimension 2 is similar to the norm form of a separable quadratic field extension $L/K$, corresponding to a finite flat quadratic cover $Y \to X$ between smooth proper curves. Then $q_v$ is isotropic if and only if the fiber of $Y \to X$ is split over the closed point corresponding to $v \in \Omega$. Hence $q_v$ is isotropic for all $v \in \Omega$ if and only if $Y \to X$ is étale (indeed, $k$ is algebraically closed). The Riemann–Hurwitz formula implies that this is only possible if the genus of $X$ is positive. We conclude that the local-global principle holds over $K$ if and only if $X = \mathbb{P}^1$. Third, there is a similar situation when $K$ is the function field of a smooth proper curve $X$ over a complete discretely valued field $k$. In this case, the local-global principle holds when $X = \mathbb{P}^1$, fails in general for quadratic forms of dimension 2 over higher genus curves, and holds for forms of dimension $\geq 3$, by the results of Colliot-Thélène, Parimala, and Suresh [7] using the patching techniques of Harbater, Hartmann, and Krashen [8].

It is the second example above that we generalize to higher dimension.

**Theorem 1.** *Let $k$ be an algebraically closed field of characteristic not 2 and $K$ the function field of a surface $X$ over $k$. Then there are counterexamples to the local-global principle for quadratic forms of dimension 4 over $K$.*

We remark that $K$ is a $C_2$-field, hence all quadratic forms of dimension $\geq 5$ are isotropic. Earlier, there were known counterexamples to the local-global principle over special classes of surfaces yet the question was still open for rational surfaces.

These counterexamples arise as an application of classification results for quadratic forms of dimension 4. Given a nondegenerate quadratic form $q$ of dimension 4 over a field $k$ (of any characteristic), the even Clifford algebra $C_0(q)$ is a quaternion algebra over the *discriminant extension*, which is an étale quadratic $k$-algebra $l$. Similar quadratic forms yield isomorphic even Clifford algebras. Conversely, given a quaternion algebra $A$ over $l$, which has trivial corestriction to $k$, there is an associated similarity class $q_{A/l/k}$ of quadratic forms of dimension 4 over $k$, called the *norm form*. In fact, the even Clifford algebra and norm form define inverse bijections between the set of similarity classes of nondegenerate quadratic forms of dimension 4 with discriminant extension $l/k$ and the set of isomorphism classes of quaternion algebras over $l$ with trivial corestriction to $k$, see [10, IV.15.B].

This has been generalized to a classification of regular quadratic forms of dimension 4 over affine schemes by Knus, Parimala, and Sridharan [9], and more generally, regular line bundle-valued quadratic forms of dimension 4 in [2, §5.3], in terms of Azumaya quaternion algebras $A$ over étale quadratic covers $Y \to X$. A *line bundle-valued quadratic form* $(E, q, L)$ over a scheme $X$ is the datum of a locally free $\mathscr{O}_X$-module $E$ of finite rank, an invertible sheaf $L$, and a quadratic form $q : E \to L$. The even Clifford algebra $C_0(E, q, L)$ was defined by Bichsel and Knus [5]. The notion of similarity is replaced by *projective similarity*, which allows for scaling by global units as well as tensoring by invertible modules.

We generalize these classification results to the degenerate context. Let $X$ be an integral scheme with 2 invertible and $D \subset X$ a divisor. A line bundle-valued quadratic form $(E, q, L)$ has *simple degeneration* along $D$ if its restriction to $X \smallsetminus D$ is regular and if for each point $x$ of $D$, the quadratic form $q \otimes_{\mathscr{O}_X} \mathscr{O}_{X,x}$ has discriminant in $\mathfrak{m}_{X,x} \smallsetminus \mathfrak{m}_{X,x}^2$ and contains a regular subform of codimension 1. If $X$ is regular, then the center of $C_0(E, q, L)$ defines the finite flat quadratic *discriminant cover* $Y \to X$. If $(E, q, L)$ has simple degeneration and even dimension, then $C_0(E, q, L)$ becomes an Azumaya algebra over $Y$, a result of Kuznetsov [11, Prop. 3.13]. Our main construction is, given an Azumaya quaternion algebra $A$ over $Y$, a line bundle-valued norm form $q_{A/Y/X}$ of dimension 4 over $X$.

**Theorem 2** ([3])**.** *Let $X$ be a regular integral scheme of dimension $\leq 2$ with 2 invertible and $Y \to X$ a finite flat quadratic cover with regular branch divisor $D$. Then the even Clifford algebra and norm form define inverse bijections between the set of projective similarity classes of quadratic forms $(E, q, L)$ of dimension 4 with simple degeneration and discriminant cover $Y \to X$ and the set of isomorphism classes of Azumaya quaternion algebras over $Y$ having split norm to $X$.*

We now review the key ingredients of the proof. The first is a norm (or corestriction) map for Azumaya algebras with respect to finite flat covers of schemes of dimension $\leq 2$. Our construction uses Zariski patching techniques of Ojanguren, relying on results of Colliot-Thélène and Sansuc [6, §2]. For the Brauer group, such a norm map was defined in greater generality by Deligne in SGA 4, Exp. 17, §6.2. Second, we prove the smoothness of the nonreductive special orthogonal group scheme $\mathrm{SO}(E, q, L)$ over $X$ associated to a quadratic form with simple degeneration, which allows to extend the exceptional isomorphisms of type ${}^2\mathsf{A}_1 = \mathsf{D}_2$ to this context. Third, we prove the Grothendieck–Serre conjecture for such special orthogonal (and projective) group schemes over discrete valuation rings. The proof then proceeds by patching the classical norm form (for étale quadratic covers) over $X \smallsetminus D$ with suitably chosen quadratic form models having simple degeneration over the local rings of generic points of components of $D$.

Finally, to construct counterexamples to the local-global principle for the function field $K$ of a smooth proper surface $X$ over an algebraically closed field $k$, we have two cases. First, if ${}_2\mathrm{Br}(X) \neq 2$, then a given 2-torsion Brauer class $\alpha$ has a quaternion algebra representative by the "period = index" result of Artin [1]. By "purity for division algebras" for schemes of dimension $\leq 2$, a result going back to Auslander and Goldman [4], there exists an Azumaya quaternion algebra $A$ on $X$ whose generic fiber is $\alpha$. Then the reduced norm $\mathrm{Nrd} : A \to \mathscr{O}_X$ is a locally isotropic quadratic form by Tsen's theorem, yet is anisotropic over $K$. Second, in the case when ${}_2\mathrm{Br}(X) = 0$, we utilize our results. We prove a geometric lemma showing that there always exists a finite flat quadratic cover $Y \to X$ between smooth surfaces, having smooth branch divisor, such that ${}_2\mathrm{Br}(Y) \neq 0$. Then as in the previous case, there exists a nonsplit Azumaya quaternion algebra $A$ over $Y$, which now has split norm to $X$ by our hypothesis in this case. Since the norm form $q_{A/Y/X}$ of dimension 4 has simple degeneration, it contains a regular subform of rank 3, hence is locally isotropic by Tsen's theorem. Finally, the norm form is anisotropic over $K$ since its even Clifford algebra gives back the nonsplit algebra $A$ over $Y$, appealing to the fact that a quadratic form of rank 4 is isotropic if and only if its even Clifford algebra is split over the discriminant extension.

Similar considerations can lead to counterexamples to the local-global principle for quadratic forms of dimension 4 over function fields of curves over totally imaginary number fields. An important open question remains: does the local-global principle hold for quadratic forms of dimension $\geq 5$ over such fields?

## References


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